Lecture 1

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CMPT 307

Data Structures & Algorithms

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(No class this Wed, May 8.)

What is this course about?

Problem
Given: Instance
Want: Solution

→

Algorithm
Given: Input instance
Finds: Solution

Various Methods for Designing algorithms

Example: Stable Marriage Problem

Amy
Bertha
Clare } n women

Xavier
Yancey
Zeus } n men

- Each person ranks the people of the opposite sex
- Want a perfect matching
- Want total ~~happiness~~ stability

Stable matching problem

Goal. Given a set of n men and a set of n women, find a "suitable" matching.

- Participants rank members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

men's preference list

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

women's preference list

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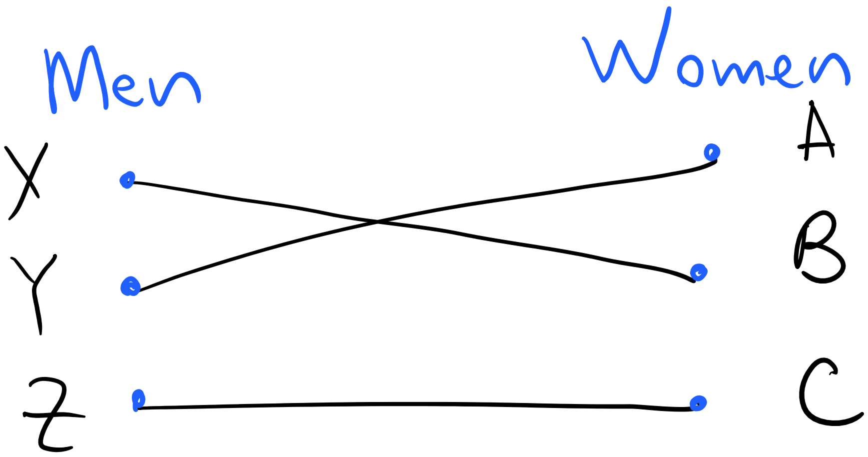

Matching : set of disjoint couples

Perfect matching : everyone is matched

Stability ?

Unstable pair (m, w) :

Not a matched couple BUT

- m prefers w to his match, &
- w prefers m to her match.

Unstable pair

Def. Given a perfect matching S, man m and woman w are unstable if:

- m prefers w to his current partner.
- w prefers m to her current partner.

Key point. An unstable pair m–w could each improve partner by joint action.

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Bertha and Xavier are an unstable pair

Stable matching problem

Def. A stable matching is a perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching (if one exists).

- Natural, desirable, and self-reinforcing condition.
- Individual self-interest prevents any man–woman pair from eloping.

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

a perfect matching S = { X–A, Y–B, Z–C }

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Stable roommate problem

Q. Do stable matchings always exist?
A. Not obvious a priori.

Stable roommate problem.

- $2n$ people; each person ranks others from 1 to $2n - 1$.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

no perfect matching is stable

A–B, C–D $\Rightarrow$ B–C unstable

A–C, B–D $\Rightarrow$ A–B unstable

A–D, B–C $\Rightarrow$ A–C unstable

Observation. Stable matchings need not exist for stable roommate problem.

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Dating Game

In each round,

- a free man M proposes to the top woman W on his list not proposed to before
- woman W evaluates the proposal:
  - if W is free, she says Yes
  - if W is engaged to M', then if M is preferable to M', then W accepts M & frees M'.

Gale-Shapley deferred acceptance algorithm

An intuitive method that guarantees to find a stable matching.

GALE–SHAPLEY (*preference lists for men and women*)

INITIALIZE S to empty matching. ← O(1)

WHILE (some man m is unmatched and hasn't proposed to every woman) ← O(1)

  w ← first woman on m's list to whom m has not yet proposed.

  IF (w is unmatched) ← O(1)

    Add pair m–w to matching S. ← O(1)

  ELSE IF (w prefers m to her current partner m') ← inverse[m] vs. inverse[m']

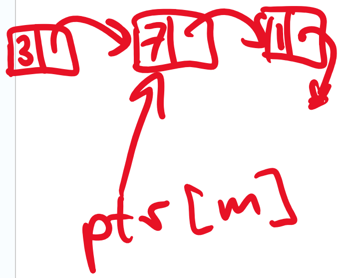
    Remove pair m'–w from matching S.

    Add pair m–w to matching S. } O(1)

  ELSE

    w rejects m. } O(1)

RETURN stable matching S.



Proof of correctness: termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of correctness: perfection

Claim. In Gale-Shapley matching, all men and women get matched.
Pf. [by contradiction]

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of GS algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of correctness: stability

Claim. In Gale-Shapley matching, there are no unstable pairs.

Pf. Suppose the GS matching S^* does not contain the pair A–Z.

- Case 1: Z never proposed to A.
  - $\Rightarrow$ Z prefers his GS partner B to A. ← men propose in decreasing order of preference
  - $\Rightarrow$ A–Z is stable.
- Case 2: Z proposed to A.
  - $\Rightarrow$ A rejected Z (right away or later)
  - $\Rightarrow$ A prefers her GS partner Y to Z. ← women only trade up
  - $\Rightarrow$ A–Z is stable.
- In either case, the pair A–Z is stable. ▪

$A - Y$

$B - Z$

$\vdots$

Gale-Shapley matching S*

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Theorem. [Gale-Shapley 1962] The Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive q acceptances, it will generally have to offer to admit more than q applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how he ranks the colleges to which he has applied; even if this is known it will not be known (c) which of the other colleges will offer to admit him. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota, and be reasonably close to the attainable optimum in quality.

Efficient implementation

Efficient implementation. We describe an $O(n^2)$ time implementation.

Input size: $O(n^2)$

Representing men and women.

- Assume men are named $1, \ldots, n$.
- Assume women are named $1', \ldots, n'$.

Representing the matching.

- Maintain a list of free men (in a stack or queue).
- Maintain two arrays $wife[m]$ and $husband[w]$.
  - if m matched to w, then $wife[m] = w$ and $husband[w] = m$
    set entry to 0 if unmatched

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- For each man, maintain a pointer to woman in list for next proposal.

Efficient implementation (continued)

Women rejecting/accepting.

- Does woman w prefer man m to man m' ?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after $O(n)$ preprocessing.

	1st	2nd	3rd	4th	5th	6th	7th	8th
pref[]	8	3	7	1	4	5	6	2

	1	2	3	4	5	6	7	8
inverse[]	4th	8th	2nd	5th	6th	7th	3rd	1st

woman prefers man 3 to 6
since inverse[3] < inverse[6]

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for i = 1 to n
   inverse[pref[i]] = i
```

Understanding the solution

For a given problem instance, there may be several stable matchings.

- Do all executions of GS algorithm yield the same stable matching?
- If so, which one?

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

an instance with two stable matching: M = { A–X, B–Y, C–Z } and M' = { A–Y, B–X, C–Z }

Understanding the solution

Def. Woman w is a valid partner of man m if there exists some stable matching in which m and w are matched.

Ex.

- Both Amy and Bertha are valid partners for Xavier.
- Both Amy and Bertha are valid partners for Yancey.
- Clare is the only valid partner for Zeus.

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

an instance with two stable matching: M = { A–X, B–Y, C–Z } and M' = { A–Y, B–X, C–Z }

Understanding the solution

Def. Woman w is a valid partner of man m if there exists some stable matching in which m and w are matched.

Man-optimal assignment. Each man receives best valid partner.

- Is it perfect?
- Is it stable?

Claim. All executions of GS yield man-optimal assignment.
Corollary. Man-optimal assignment is a stable matching!

Man optimality

Claim. GS matching S^* is man-optimal.
Pf. [by contradiction]

- Suppose a man is matched with someone other than best valid partner.
- Men propose in decreasing order of preference
  $\Rightarrow$ some man is rejected by valid partner during GS.
- Let Y be first such man, and let A be the first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected by A in GS, A forms (or reaffirms) engagement with a man, say Z.
  $\Rightarrow$ $\boxed{A \text{ prefers } Z \text{ to } Y.}$
- Let B be partner of Z in S.
- Z has not been rejected by any valid partner (including B) at the point when Y is rejected by A. ⟵ because this is the first rejection by a valid partner
- Thus, Z has not yet proposed to B when he proposes to A.
  $\Rightarrow$ $\boxed{Z \text{ prefers } A \text{ to } B.}$
- Thus A–Z is unstable in S, a contradiction. ▪

$A - Y$
$B - Z$
$\vdots$

stable matching S

Woman pessimality

Q. Does man-optimality come at the expense of the women?
A. Yes.

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S^*.
Pf. [by contradiction]

- Suppose A–Z matched in S^* but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
  $\Rightarrow$ $\boxed{A \text{ prefers } Z \text{ to } Y.}$
- Let B be the partner of Z in S. By man-optimality, A is the best valid partner for Z.
  $\Rightarrow$ $\boxed{Z \text{ prefers } A \text{ to } B.}$
- Thus, A–Z is an unstable pair in S, a contradiction. ▪

$A - Y$
$B - Z$
$\vdots$

stable matching S

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Deceit: Machiavelli meets Gale-Shapley

Q. Can there be an incentive to misrepresent your preference list?

- Assume you know men's propose-and-reject algorithm will be run.
- Assume preference lists of all other participants are known.

Fact. No, for any man; yes, for some women.

men's preference list

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

women's preference list

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Amy lies

	1st	2nd	3rd
A	Y	Z	X
B	X	Y	Z
C	X	Y	Z

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Extensions: matching residents to hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling
to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy. ← hospital X wants to hire 3 residents

Def. Matching is S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- Either r is unmatched, or r prefers h to her assigned hospital; and
- Either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Historical context

National resident matching program (NRMP).

- Centralized clearinghouse to match med-school students to hospitals.
- Began in 1952 to fix unraveling of offer dates.
- Originally used the "Boston Pool" algorithm.
- Algorithm overhauled in 1998.
  - med-school student optimal
  - deals with various side constraints (e.g., allow couples to match together) ← stable matching is no longer guaranteed to exist
- 38,000+ residents for 26,000+ positions.

hospitals began making
offers earlier and earlier,
up to 2 years in advance

The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design

By Alvin E. Roth and Elliott Peranson*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains this; that each applicant interviews only a small fraction of available positions is important. We also describe engineering aspects of the design process. (JEL C78, B41, J44)

2012 Nobel Prize in Economics

Lloyd Shapley. Stable matching theory and Gale-Shapley algorithm.

Alvin Roth. Applied Gale-Shapley to matching new doctors with hospitals, students with schools, and organ donors with patients.

Lloyd Shapley   Alvin Roth

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Social Implications: Fairness

GS algo is unfair to women (or to men)

Q: How to define "fairness"?

Q: Is there an efficient algo to find a "fair" stable matching?

Algorithms influence our everyday lives! We need to take care.

4 Kinds of Algorithms

4 Kinds of Algorithms

Obvious : by problem definition

Methodical : by a method good for a class of problems

Clever : specific to a problem

Miraculous : how did anyone think of this?

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