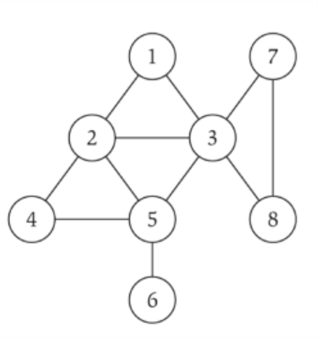
Undirected graphs

Notation. $G = (V, E)$

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.



$V = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$E = \{ 1\text{-}2, 1\text{-}3, 2\text{-}3, 2\text{-}4, 2\text{-}5, 3\text{-}5, 3\text{-}7, 3\text{-}8, 4\text{-}5, 5\text{-}6, 7\text{-}8 \}$

$m = 11, n = 8$

3

Some graph applications

graph	node	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

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Graph representation: adjacency matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

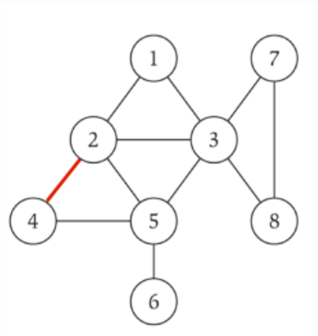
- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.

Ω : $\geq$

O : $\leq$

Θ : $=$ $n \times n$

$O(n^2)$



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

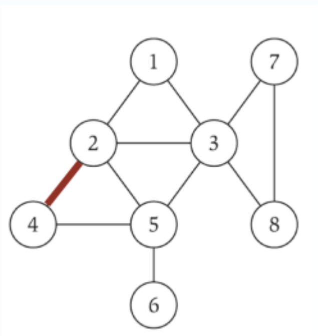
Graph representation: adjacency lists

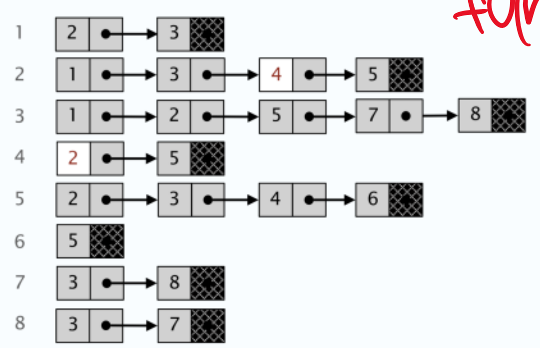
Adjacency lists. Node indexed array of lists.

- Two representations of each edge.
- Space is $\Theta(m + n)$.
- Checking if (u, v) is an edge takes $O(degree(u))$ time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of u

$$\text{size} = \sum_{v \in V} \deg(v) = 2 \cdot m + O(n) = O(m+n)$$



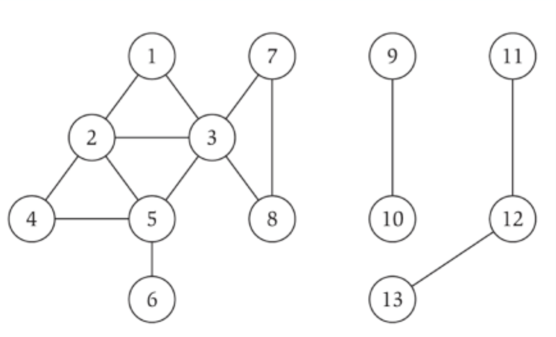


Paths and connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair v_{i-1}, v_i is joined by an edge in E.

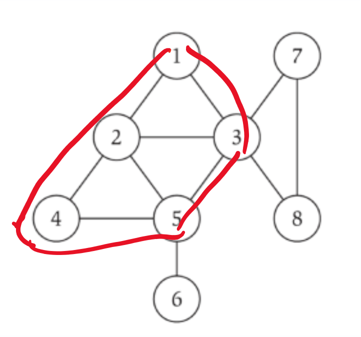
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k - 1$ nodes are all distinct.



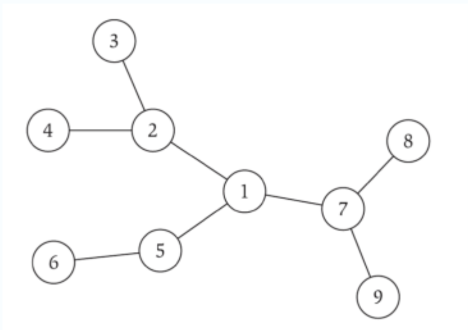
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

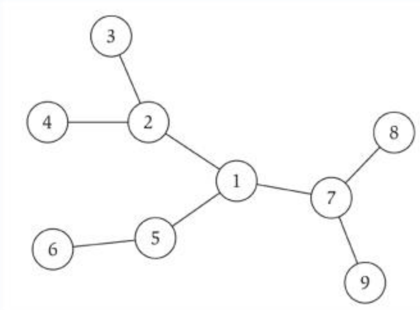
- G is connected.
- G does not contain a cycle.
- G has $n-1$ edges.



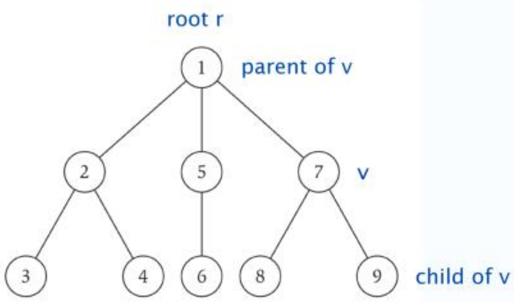
Rooted trees

Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



a tree



the same tree, rooted at 1

Tree on n nodes
has n-1 edges.

Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

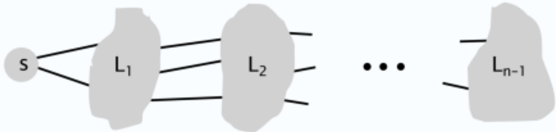
s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Applications.

- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.

Breadth-first search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



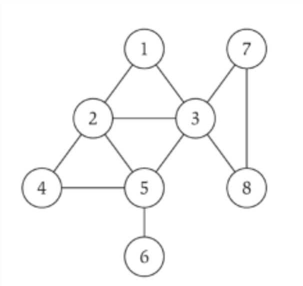
BFS algorithm.

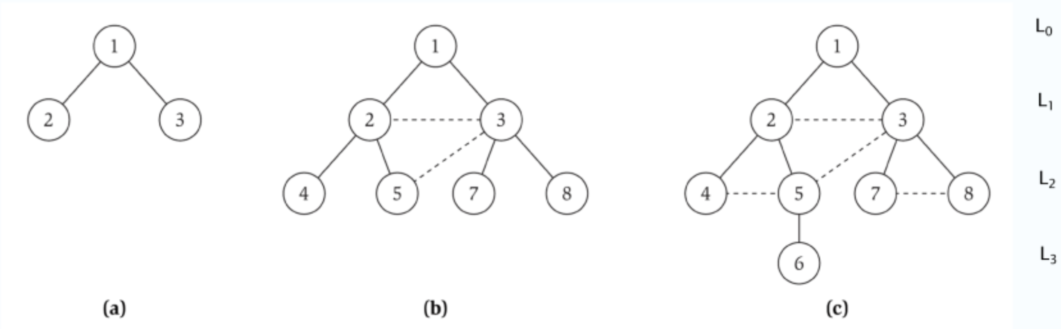
- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

Breadth-first search

Property. Let T be a BFS tree of $G = (V, E)$, and let (x, y) be an edge of G. Then, the level of x and y differ by at most 1.





(a) (b) (c)

Breadth-first search: analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

linear time

size $\Theta(m+n)$

Pf.

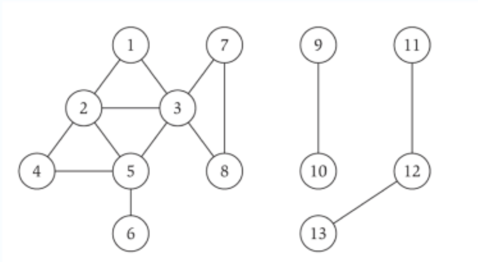
- Easy to prove $O(n^2)$ running time:
  - at most n lists $L[i]$
  - each node occurs on at most one list; for loop runs $\leq n$ times
  - when we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge

- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are $degree(u)$ incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V}\ degree(u) = 2m$. ▪

each edge (u, v) is counted exactly twice in sum: once in degree(u) and once in degree(v)

Connected component

Connected component. Find all nodes reachable from s.

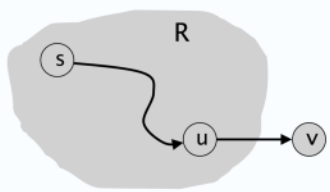


Connected component containing node $1 = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$.

Connected component

Connected component. Find all nodes reachable from s.

```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```



it's safe to add v

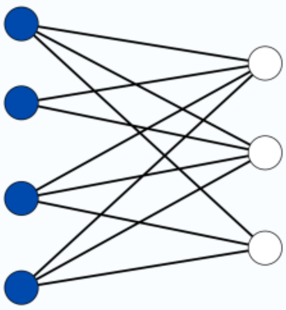
Theorem. Upon termination, R is the connected component containing s.

- BFS = explore in order of distance from s.
- DFS = explore in a different way.

Bipartite graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored blue or white such that every edge has one white and one blue end.

Applications.
- Stable marriage: men = blue, women = white.
- Scheduling: machines = blue, jobs = white.

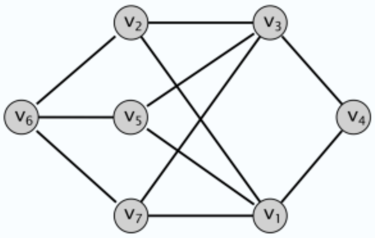


a bipartite graph

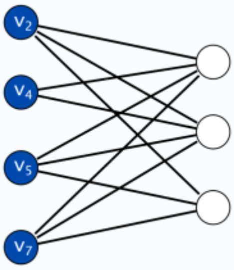
Testing bipartiteness

Many graph problems become:
- Easier if the underlying graph is bipartite (matching).
- Tractable if the underlying graph is bipartite (independent set).

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

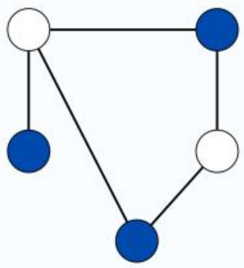


another drawing of G

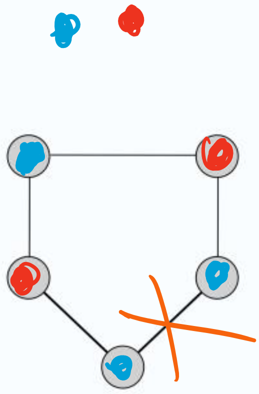
An obstruction to bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

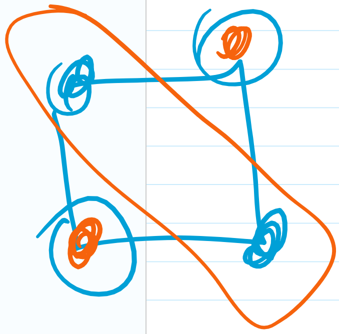
Pf. Not possible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)



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Bipartite graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

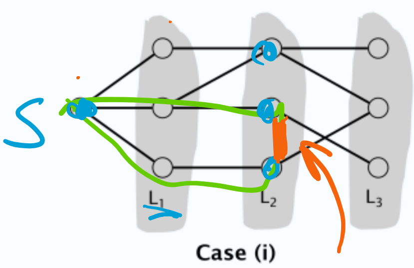
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

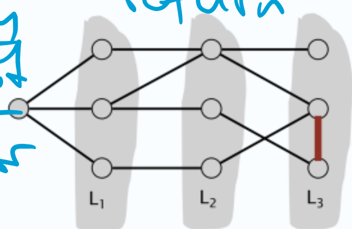
Check:

Color [i] = 0 or 1

for e = (i,j) ∈ E
  if color [i] = color [j]
  then return "Non-bipartite"
  endif
endfor



Case (i)



Case (ii)

Time:
(1) $O(m+n)$: BFS
(2) $O(m)$

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algo 2-Colour

Given $G = (V, E)$ (connected)

Given $G = (V, E)$ (connected)

1. Run BFS on G,
   getting layers $L_0, L_1, L_2, \dots$
2. For every node $v \in V$,
   color v 0 if v is in even layer L_{2i}
   or 1 if v is in odd layer L_{2i+1}
3. Check if any edge $(u,v) \in E$ has $\text{Color}(u) = \text{Color}(v)$
   If so, output "Not Bipartite".
   Otherwise, output "Bipartite".

Runtime Analysis: $O(m+n)$

(1) BFS : time $O(m+n)$
(2) assigning colors to nodes : $O(n)$
(3) Checking for monochromatic edges: $O(m)$

Correctness Analysis:

✓ (1) G not bipartite $\Rightarrow$ algo says "Non-bip."

(2) G bipartite $\Rightarrow$ algo says "Bipartite"

$\equiv$

G is not bipartite $\Leftarrow$ algo says "Non-bipartite"

We will argue that when the algo says

We will argue that when the algo says "Non-bipartite" on an input graph G, then G contains an odd cycle. Hence, G indeed is not bipartite.

Bipartite graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

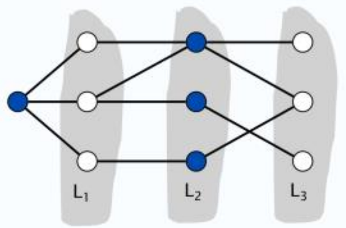
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Case 1: no edges in same layer.

Pf. (i)

- Suppose no edge joins two nodes in same layer.
- By BFS property, each edge join two nodes in adjacent levels.
- Bipartition: white = nodes on odd levels, blue = nodes on even levels.



Case (i)

This algo 2-colors the graph unless it has an odd cycle.

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Bipartite graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

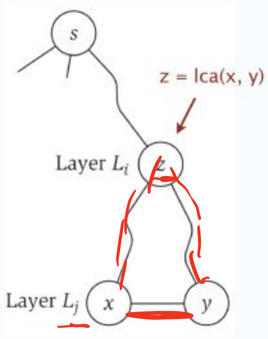
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Case 2: ∃ edge in same layer

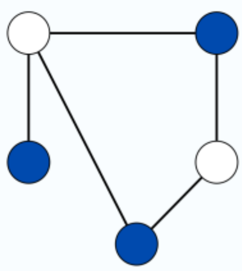
Pf. (ii)

- Suppose (x, y) is an edge with x, y in same level L_j.
- Let $z = lca(x, y)$ = lowest common ancestor.
- Let L_i be level containing z.
- Consider cycle that takes edge from x to y, then path from y to z, then path from z to x.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd. ▪

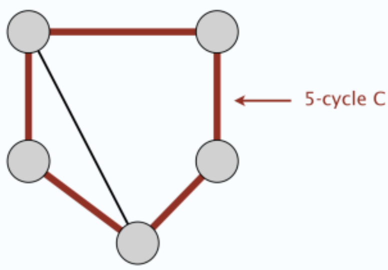


The only obstruction to bipartiteness

Corollary. A graph G is bipartite iff it contain no odd length cycle.



bipartite
(2-colorable)

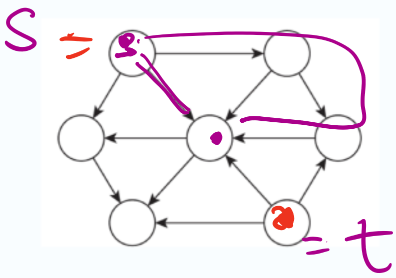


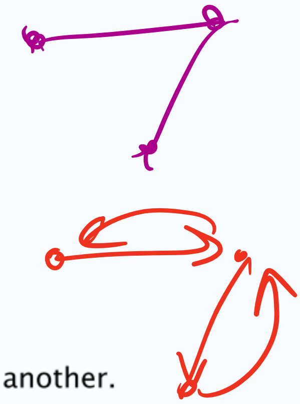
not bipartite
(not 2-colorable)

Directed graphs

Notation. $G = (V, E)$.

- Edge (u, v) leaves node u and enters node v.





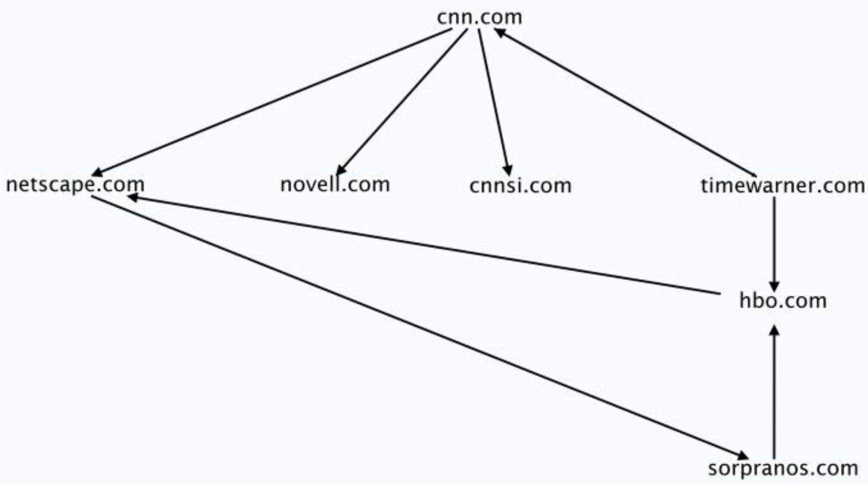
Ex. Web graph: hyperlink points from one web page to another.

- Orientation of edges is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

World wide web

Web graph.

- Node: web page.
- Edge: hyperlink from one page to another (orientation is crucial).
- Modern search engines exploit hyperlink structure to rank web pages by importance.



Some directed graph applications

directed graph	node	directed edge
transportation	street intersection	one-way street
web	web page	hyperlink
food web	species	predator-prey relationship
WordNet	synset	hypernym
scheduling	task	precedence constraint
financial	bank	transaction
cell phone	person	placed call
infectious disease	person	infection
game	board position	legal move
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

Graph search

Directed reachability. Given a node s, find all nodes reachable from s.

u

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path from s and t?

Graph search. BFS extends naturally to directed graphs.

& DFS

Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Strong connectivity

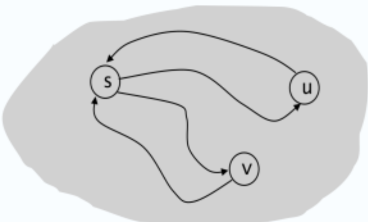
Def. Nodes u and v are mutually reachable if there is a both path from u to v and also a path from v to u.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.
Pf. $\Leftarrow$ Path from u to v: concatenate $u \leadsto s$ path with $s \leadsto v$ path.
Path from v to u: concatenate $v \leadsto s$ path with $s \leadsto u$ path. ▪

ok if paths overlap



Strong connectivity: algorithm

Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in $G^{reverse}$.
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪

reverse orientation of every edge in G

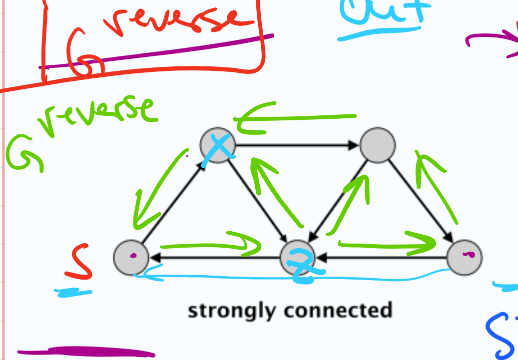
Check s is a hub.

Run time $O(m+n)$

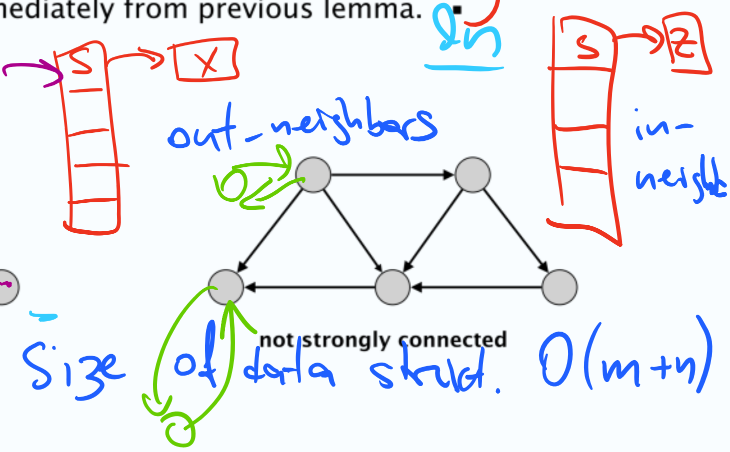
G reverse

Out

In



strongly connected

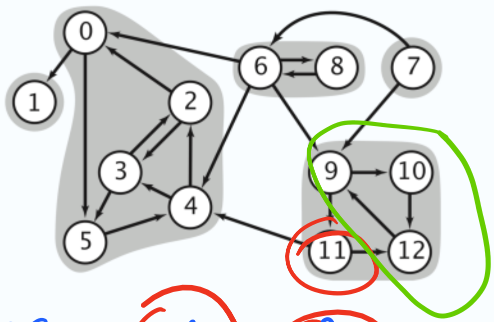


not strongly connected

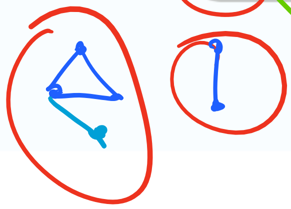
Size of data struct. $O(m+n)$

Strong components

Def. A strong component is a maximal subset of mutually reachable nodes.



Undir. graphs



Theorem. [T ... n $O(m + n)$ time.

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DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

to find all strong components

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Depth-First Search (DFS)

DFS (G = (V, E))
  for each v ∈ V, mark v not explored
  endfor

  for each v ∈ V
    if v is not explored
    then DFS-Visit (v)
    endif
  endfor

DFS-Visit (u)
  mark u explored
  for each edge u — v
    if v not explored

for each edge

if v not explored

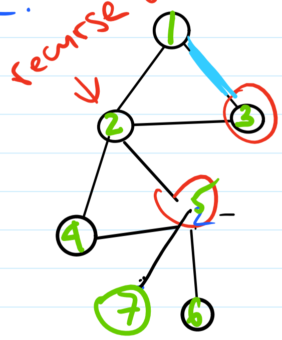
then DFS-Visit(v)

endif

endfor

Ex. DFS

recurse



DFS tree

DFS tree T

Observe:

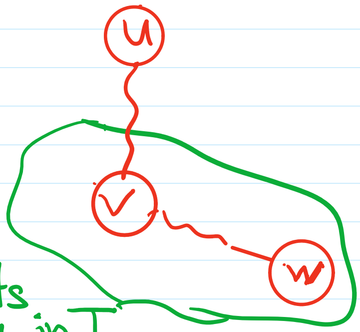
DFS-Visit(u)

v explored

w explored

exit DFS-Visit(u)

time



descendants of u in T

DFS Tree Property:

T DFS tree of $G=(V,E)$

$(x,y) \in E$ BUT not edge of T

Then one of x, y is an ancestor

of the other in T.

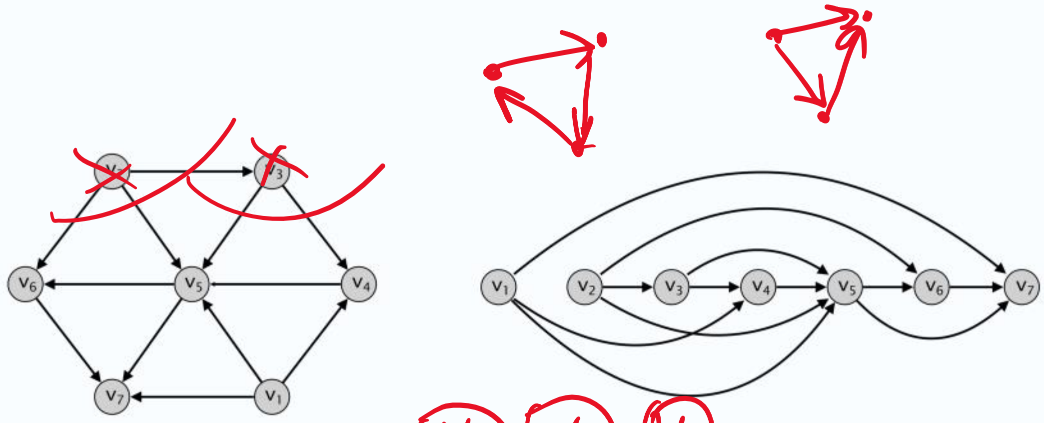
Proof: Say DFS(x) is called before y was discovered

Then y is discovered while still within the DFS(x) call. Hence, $x \rightsquigarrow y$ in the DFS tree.

Directed acyclic graphs

Def. A DAG is a directed graph that contains no directed cycles.

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG

v_2 v_3 v_1 . . .

a topological ordering

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Precedence constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications.

- Course prerequisite graph: course v_i must be taken before v_j.
- Compilation: module v_i must be compiled before v_j. Pipeline of computing jobs: output of job v_i needed to determine input of job v_j.

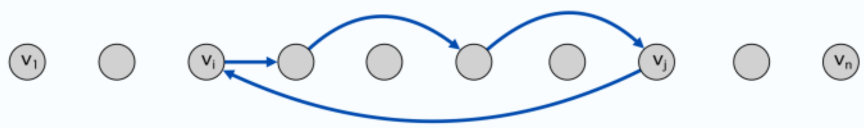
Directed acyclic graphs

Lemma. If G has a topological order, then G is a DAG.

Pf. [by contradiction]

- Suppose that G has a topological order $v_1, v_2, \ldots, v_n$ and that G also has a directed cycle C. Let's see what happens.
- Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.
- By our choice of i, we have $i < j$.
- On the other hand, since (v_j, v_i) is an edge and $v_1, v_2, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪

the directed cycle C



the supposed topological order: $v_1, \ldots, v_n$

Directed acyclic graphs

Lemma. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

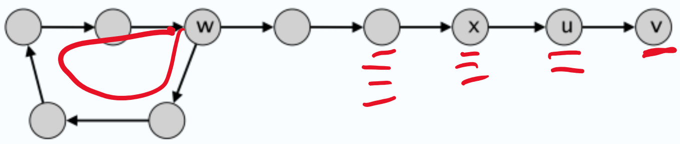
Q. If so, how do we compute one?

Directed acyclic graphs

Lemma. If G is a DAG, then G has a node with no entering edges.

Pf. [by contradiction]

- Suppose that G is a DAG and every node has at least one entering edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one entering edge (u, v) we can walk backward to u.
- Then, since u has at least one entering edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪



Directed acyclic graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. [by induction on n]

- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node v with no entering edges.
- $G - \{ v \}$ is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, $G - \{ v \}$ has a topological ordering.
- Place v first in topological ordering; then append nodes of $G - \{ v \}$
- in topological order. This is valid since v has no entering edges. ▪

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To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
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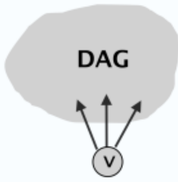
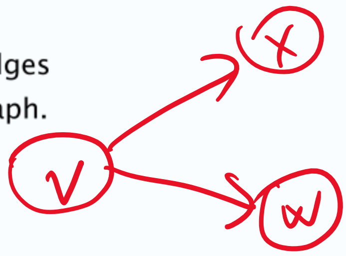


Topological sorting algorithm: running time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

- Maintain the following information:
  - $count(w)$ = remaining number of incoming edges
  - S = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete v
  - remove v from S
  - decrement $count(w)$ for all edges from v to w; and add w to S if $count(w)$ hits 0
  - this is $O(1)$ per edge ▪



BFS & DFS Implementations

- BFS: use queues
- DFS: use stacks

Algo BFS: $G = (V, E)$, $s \in V$

for each $v \in V$
  Explored $[v]$ = False
endfor
Explored $[s]$ = True
$L[0] = \langle s \rangle$
$i = 0$
while $L[i] \neq \emptyset$
  $L[i+1] = \emptyset$
  for each $u \in L[i]$
    for each edge $(u, v) \in E$
      if Explored $[v]$ = False
      then Explored $[v]$ = True
        $L[i+1] = L[i+1] + \langle v \rangle$; $T = T + (u, v)$
      endif
    endfor
  endfor
  $i = i + 1$
endwhile

$T = \emptyset$

algo DFS (s)

stack $S = \langle s \rangle$ (set Explored[v] = False, $\forall v \in V$)

while $S \neq \emptyset$

u = pop(S)

if Explored(u) = False

then Explored(u) = True, T = T + (parent(u), u)

for each edge $(u, v) \in E$

push(S, v); parent(v) = u

endfor

endif

endwhile