Decidability & Semi-Decidability for NTMs

Computation tree on input $x$

Configuration:
- Next state of $Q$
- The contents of all tapes
- Positions of tape head on all tapes

Computation of a TM on input $x$: a sequence of configurations
$c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow \ldots \rightarrow c_t \rightarrow \ldots$

where
$c_1$ = initial config.
$c_i \rightarrow c_{i+1}$ follows from $c_i$ using $S$ of TM.

$L$ is decidable by an NTM $M$ if
$\forall x \in L \Rightarrow M$ on $x$ accepts
(on some branch)
∀ x ∈ L \Rightarrow M\text{ on } x\text{ accepts (on some branch)}

∀ x \notin L \Rightarrow M\text{ on } x\text{ rejects on all branches} \cup \text{halt (semi)}

Claim: \text{L decidable by an NTM} \quad \iff \quad \text{L decidable by a DTM.}

\text{BFS-style exploration & check if all leaves are rejects.}

\text{L is semi-decidable by an NTM } M \text{ if}

∀ x ∈ L \Rightarrow M\text{ on } x\text{ accepts (on some branch)}

∀ x \notin L \Rightarrow M\text{ on } x\text{ does not accept (could be inf. branches)} \cup \text{halt (semi)}

\text{NTM decidability} \equiv \text{L (semi)
\[ \text{DTM (semidecidable)} \]

Claim: \( L \) semi-decidable
\[ \Rightarrow \quad L = \text{poly}^{-1} - L \quad \text{semi-decidable} \]
\[ \Rightarrow \quad L \quad \text{decidable.} \]

\[ \begin{array}{c}
X \\
\Rightarrow \\
\rightarrow \\
\text{TA}_L \\
\downarrow \\
\rightarrow \\
\text{A}_L \\
\rightarrow \\
\text{Yes} \\
\end{array} \quad \begin{array}{c}
\text{Yes} \quad \text{(x \in L)} \\
\text{No} \quad \text{(x \notin L)} \\
\end{array} \]

Cor.: \( \overline{A}_{TM} \) is not semi-decidable.

Pf.: \( A_{TM} \) is semi-decidable.
\[ = \{ \langle M, w \rangle \mid TM \text{ accepts } w \} \]

On the other hand, we know
\[ A_{TM} \text{ is not decidable.} \]

Thus, (1) The class of decidable lang's
is closed under complementation.
(2) The class of semi-decidable lang's
(2) The class of semi-decidable lang's is NOT closed under complement.

**Proof:**
(1) \( q_{acc} \leftrightarrow q_{reg} \)  \( \checkmark \)
(2) \( A_{TM} \)  \( \checkmark \) \( \Box \)

**Examples of undecidable problems**

\[
E_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}
\]

is undecidable.

**Proof:** Suppose \( E_{TM} \) is decidable.
Then can decide \( A_{TM} \).

\[<M, w> \]

1. Construct \( M' \): "On input \( x \), simulate \( M \) on \( w \).
   If \( M \) acc \( w \), then accept."

2. Check if \( \langle M' \rangle \notin E_{TM} \)

\( (1) \) \( M \) acc \( w \) \( \Rightarrow \) \( L(M') = \Sigma^* \neq \emptyset \)

\( (2) \) \( M \) not acc \( w \) \( \Rightarrow \) \( L(M') = \emptyset \)

...
\[ \text{ALL}_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^+ \} \]
is undecidable.

\[ \text{EQ}_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \} \]
is undecidable.

**Proof:** Suppose \( \text{EQ}_{TM} \) is decidable.

Then \( \text{E}_{TM} \) is decidable.

1. Define \( M_{\phi} \): "Reject"

2. \( \langle M, M_{\phi} \rangle \in \text{EQ}_{TM} \)

\( \text{E}_{TM} \)-decider