Lecture 11

Wednesday, June 5, 2019 9:57 AM

Last time: Examples of undecidable problems

- $A_{TM} = \{ \langle M, w \rangle \mid \text{TM } M \text{ accepts } w \}$
- $E_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}$
- $ALL_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^* \}$
- $EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$

The undecidability proofs are by reductions.

They can be generalized to show

Rice's Theorem: Every non-trivial property P of TMs is undecidable.

Def (property):

a property $P = \{ \langle M \rangle \mid \text{TM } M \text{ satisfies} \ldots \}$

s.t. $\forall$ TMs M_1, M_2

$L(M_1) = L(M_2) \Rightarrow$ both $\langle M_1 \rangle, \langle M_2 \rangle \in P$,

or both $\langle M_1 \rangle, \langle M_2 \rangle \notin P$.

Ex: $P = \{ \langle M \rangle \mid L(M) = \emptyset \}$ is a property.

$\forall\ M_1, M_2,\quad L(M_1)=L(M_2) \Rightarrow$

either $\langle M_1\rangle, \langle M_2\rangle \in P$

(if $L(M_1)=\emptyset$)

or $\langle M_1\rangle, \langle M_2\rangle \notin P$

(if $L(M_1)\neq\emptyset$)

Ex: $Q = \{\langle M\rangle \mid \text{TM } M \text{ has} \leq 100 \text{ states}\}$ is NOT a property.

$\exists\ M_1$ on ≤ 100 states with $L(M_1)=\emptyset$

$\exists\ M_2$ on > 100 states with $L(M_2)=\emptyset$

So $\exists M_1, M_2,\quad L(M_1)=L(M_2)$, but

$\langle M_1\rangle \in Q \quad \& \quad \langle M_2\rangle \notin Q$

A property P is called non-trivial if $\exists\ \langle M_1\rangle \in P$, and

$\exists\ \langle M_2\rangle \notin P$.

Rice's Thm: Every non-trivial property P is undecidable.

Proof: Since P is non-trivial:

M_1 s.t. $L(M_1) = \emptyset$

Wlog, $\langle M_1 \rangle \in P$.

$\exists$ M_2 s.t. $\langle M_2 \rangle \notin P$

By contradiction: Suppose P is decidable.

Will prove A_{TM} is decidable.

$A_{TM} = \{ \langle M, w \rangle \mid$ TM M accepts $w \}$

1. Construct a TM

A: "On input x,
run M on w.
If M accepts w,
then simulate M_2 on x."

A: (1) M accepts w $\Rightarrow L(A) = L(M_2)$

$\Rightarrow \langle A \rangle \notin P$

(2) M does not accept w $\Rightarrow L(A) = \emptyset$

$\Rightarrow \langle A \rangle \in P$

2. Run a decider for P on $\langle A \rangle$.

If it accepts $\langle A \rangle$ $(\langle A \rangle \in P)$, then answer M not acc w.

If it rejects $\langle A \rangle$ $(\langle A \rangle \notin P)$, then answer M acc w.

Reductions

Def: A language A is reducible to B $(A \leq B)$ if $\exists$ a computable function $f : \Sigma^* \to \Sigma^*$ s.t.

$\forall\, x \in \Sigma^*$,

$$x \in A \iff f(x) \in B.$$

Thm: $A \leq B$ & B semi-decidable

$\implies$ A is also semi-decidable.

not semi-dec.

Thm: $A \le B$ & A is undecid.

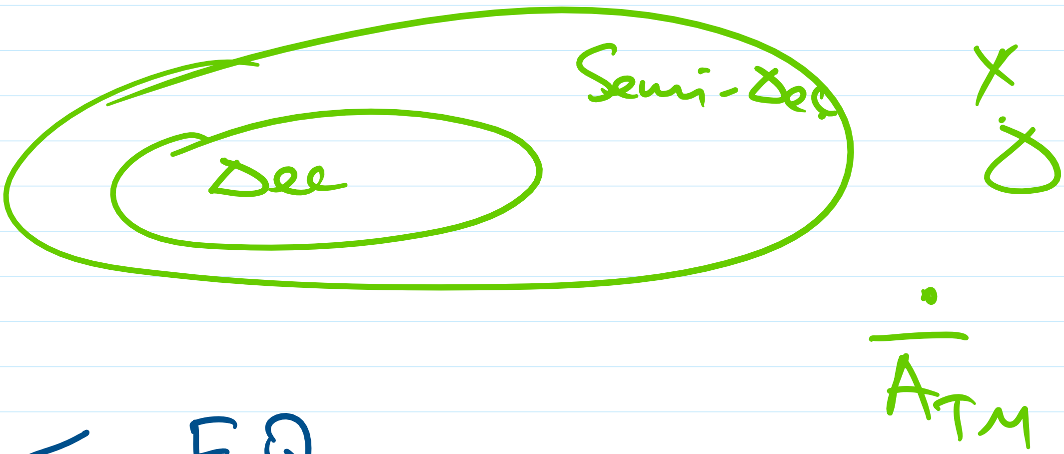
not semi-decid.

$\Rightarrow$ B is undecid.

Examples:

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$$

is not semi-decidable.



$$\overline{A_{TM}} \le EQ_{TM}$$

Want f: Computable s.t.

$\forall \langle M, w \rangle$

(1) M not acc w $\Rightarrow$ $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$ s.t. $L(M_1) = L(M_2)$

(2) M acc w $\Rightarrow$ $f(\langle M, w \rangle) =$

(2) M acc $w \Longrightarrow f(\langle M, w\rangle) = \langle M_1, M_2\rangle$ s.t. $L(M_1) \neq L(M_2)$

Build reduction f:

On input $\langle M, w\rangle$, construct M_1 : "On input x, simulate M on w."

M_2 : $L(M_2) = \emptyset$

$$L(M_1) = \begin{cases} \Sigma^* & \text{if } M \text{ acc } w \\ \emptyset & \text{if } M \text{ not acc } w \end{cases}$$

If M acc w, then $L(M_1) \neq L(M_2)$
If M not acc w, then $L(M_1) = L(M_2)$

$\square$

$INF = \{ \langle M\rangle \mid L(M) \text{ infinite} \}$ is undecidable (say, by Rice's)

is not semidecidable.

Also, $\overline{INF}$ is not semidecidable.

To prove $\overline{INF}$ not semi-dec.

$$\overline{A_{TM}} \leq \overline{INF}$$

$\overline{A_{TM}}$ not semi-dec $\Rightarrow$ $\overline{INF}$ not semi-dec.

$\langle M, w \rangle \longmapsto M'$

M acc $w \Rightarrow L(M')$ inf.

M not acc $w \Rightarrow L(M')$ finite

M' : "On input x, simulate M on w."

M acc $w \Rightarrow L(M') = \Sigma^*$ inf

M not acc $w \Rightarrow L(M') = \emptyset$ finite

Want

$$\overline{A_{TM}} \leq INF$$