Lecture 16

Monday, June 17, 2019 12:15 PM

Last time: Gödel's Incompleteness Theorem

"Any (powerful enough) proof system for arithmetic cannot be both sound and complete."

We used a computability-based proof.

$$\text{Provable} = \{ \langle \varphi \rangle \mid \varphi \text{ is provable in a given proof system} \}$$

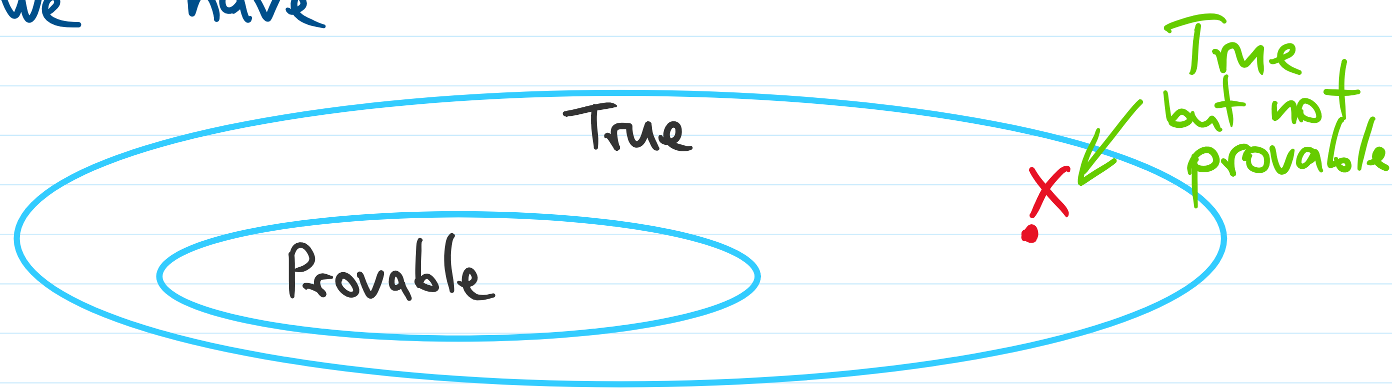
$$\text{True} = \{ \langle \varphi \rangle \mid \varphi \text{ is true over } \mathbb{N} \}$$

Claim 1: Provable is semi-decidable.

Claim 2: True is not semi-decidable.

Hence, Provable $\neq$ True.

So, for any sound proof system P, we have



Today: Another proof of the Incompleteness Theorem

Today: Gödel's proof of the Incompleteness Theorem.

We'll construct an explicit true but not provable arithmetic sentence

$$\underbrace{G \equiv \text{" } G \text{ is not provable in } P \text{ "}}_{\text{" I am not provable "}}$$

Suppose we have such a sentence G.

Then

Case 1: G is false. $\Rightarrow$

G is provable in P $\Rightarrow$

P proves a false sentence $\Rightarrow$ P not sound.

Case 2: G is true. $\Rightarrow$

G is not provable in P $\Rightarrow$

P doesn't prove a true sentence G.

So, if P is sound, then P is incomplete (can't prove G).

Gödel numbers

Every arithmetic formula (sequence of formulas) A is assigned a unique number $\varphi(A)$.

Dual role of numbers

(1) a number n is just a number n (an input for a formula $A(x)$)

(2) a number n is interpreted as the Gödel number of a formula B (s.t. $\varphi(B) = n$)

So can have sentences

$$A(n)$$

that could talk about other formulas (with Gödel number n),

formulas (with Gödel number n),
and, in particular,
could talk about themselves
(self-reference).

Define

$$\text{proof}(n, m) = \begin{cases} 1 & \text{if } n \text{ is a proof of } m \\ 0 & \text{o.w.} \end{cases}$$

This fn is computable!

as we've seen before,

TMs $\longrightarrow$ arithm. formulas simulating these TMs

Hence, $\exists$ ar. formula

$$\text{Proof}(x, y)$$

$\forall\, a, b \in \mathbb{N}$

$$\text{Proof}(a, b) \equiv T \quad \text{iff}$$

a is a proof of b

Define

$$\mathrm{Thm}(b) \equiv \exists x\, \mathrm{Proof}(x, b)$$

Want: $G \equiv \neg \mathrm{Thm}(n)$

where n is $\varphi(G)$.

$\neg \mathrm{Thm}(x)$: $n = \varphi(\neg \mathrm{Thm}(x))$

$$\neg \mathrm{Thm}(\underline{n}) \neq \neg \mathrm{Thm}(x)$$

Define

$$\mathrm{sub}(m, n) = j$$

where

Where

$m = \varphi(A(x))$ → just n

$$j = \varphi(\ A([n]) \ )$$

is Computable!

Hence, $\exists$ ar. formula

$$Sub([m],[n]) = [j]$$

iff $\quad sub(m,n) = j$

(In truth, $SUB([m],[n],[j])$

$\equiv T \iff sub(m,n) = j$)

Define

Define

1. $A(x) \equiv \neg Thm(Sub(x,x))$
2. Let $n_0 = \varphi(A(x))$ be the G. number of $A(x)$.

3. Define

$$G \equiv A([n_0])$$
$$\equiv \neg Thm(\underbrace{Sub([n_0],[n_0])}_{= m_0})$$

$A(x) \quad n_0$

$$\underline{m_0 = sub(n_0, n_0)}$$

$$m_0 = \varphi(A([n_0])) = \varphi(G)$$

Gödel's 2^{nd} Incompleteness Thm

"$\forall$ proof system P for arithmetic (powerful enough*)
If P is consistent (doesn't prove contradictions),

then P cannot prove its own consistency."