Lecture 18

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Last time: Gödel's 2^{nd} Incompleteness Thm

"If $\mathfrak{T}$ is consistent, then

$$\mathfrak{T} \nvdash \mathrm{Cons}_{\mathfrak{T}}.$$

(doesn't prove its consistency)"

Application:

Define $H \equiv$ "$\mathfrak{T} \vdash H$"

$\equiv$ "I am provable"

(can be constructed using the Recursion Thm for ar. formulas from last time)

Theorem: H is true, i.e.,

$$\mathfrak{T} \vdash H.$$

Notation: "$[\,]A$" means "$\mathfrak{T} \vdash A$"

Löb's Lemma:

If $\mathfrak{T} \vdash ([\,]A \to A)$,

then $\mathfrak{T} \vdash A$.

Proof of Thm (from Löb's Lemma):

We have $H \equiv [\,]H$, in particular,

$$[\,]H \to H.$$

Moreover, $\mathfrak{T}$ can prove $[\,]H \to H$.

By Löb's Lemma, $\mathfrak{T}$ proves H. $\square$

Proof of Löb's Lemma (from Gödel's 2^{nd} Thm).

We have $\mathfrak{T} \vdash ([\,]A \to A)$.

Want to show $\mathfrak{T} \vdash A$.

Define

$$\mathfrak{T}' = \mathfrak{T} \cup \{\neg A\}$$

$$\mathfrak{I}' = \mathfrak{I}' \cup \{\neg A\}$$

new axiom

Note $\mathfrak{I}' \vdash ([\,]A \to A)$ as well.

Since $\mathfrak{I}' \vdash \neg A$, we get

$$\mathfrak{I}' \vdash \neg [\,] A$$

$$|||$$

"A is not provable in $\mathfrak{I}$"

what about in $\mathfrak{I}'$?

<u>Claim:</u> $(\mathfrak{I} \vdash A) \iff (\mathfrak{I}' \vdash A)$

<u>Proof of Claim:</u> (1) $\mathfrak{I} \vdash A \Rightarrow \mathfrak{I}' \vdash A$

(since $\mathfrak{I}' = \mathfrak{I} \cup \{\neg A\}$)

(2) $\mathfrak{I}' \vdash A \equiv$

$\mathfrak{I} \cup \{\neg A\} \vdash A \equiv$ $\boxed{\begin{array}{l} C \vdash \delta \\ \iff \vdash \delta \vee \neg C \end{array}}$

$$\mathfrak{I} \vdash \underbrace{A \vee \neg(\neg A)}$$

$\equiv$

$A \vee A \equiv A$

$\mathfrak{T} \vdash A$. $\square$

<u>Ex.</u> "Proof by contradiction"

$$\mathfrak{T} \vdash X \iff \mathfrak{T} \cup \{\neg X\} \vdash \bot$$

$$\mathfrak{T} \vdash X \vee C \iff \mathfrak{T} \cup \{\neg X\} \vdash C$$

We conclude that

$\mathfrak{T}'$ proves that "A is <u>not</u> provable in $\mathfrak{T}'$"

<u>Note:</u> If something is not provable in $\mathfrak{T}'$, then $\mathfrak{T}'$ is <u>consistent</u>!

So, $\mathfrak{T}'$ proves that $\mathfrak{T}'$ is consistent!

By Gödel's 2nd Incompleteness Thm, $\mathfrak{T}'$ is inconsistent!

Hence,

$$\mathfrak{I} \cup \{\neg A\} \vdash \perp$$

(contradiction)

This implies

$$\mathfrak{I} \vdash \underbrace{\neg(\neg A)}_{\equiv A} . \qquad \square$$

Decidability of Presburger's Arithmetic

Have + only
(and no *)

$$\exists x \, \forall y \, \exists z \, (z = x + x + y)$$

Presb. ar. is decidable
(Truth is decidable)

Presb. ar. is both sound & complete.

Summary

(1) $\vdash, \models$: no sound & complete p.s. exist

(2) $G \equiv$ "I'm not provable"

(3) $\mathcal{F} \nvdash \mathrm{Cons}_{\mathcal{F}}$
if $\mathcal{F}$ is indeed cons.

(4) Truth of ar. sentences is not decidable
(semi-dec, arithmetic)