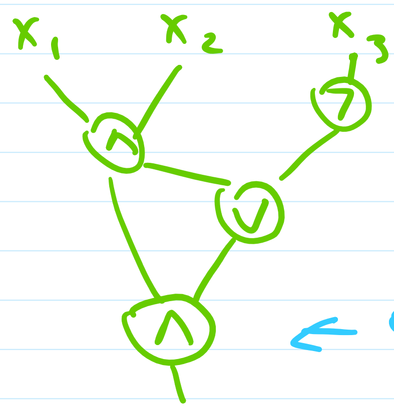
**Last time:** SAT is NP-complete.

$$SAT = \{ \langle \varphi \rangle \mid \text{propositional formula } \varphi(x_1, \dots, x_n) \text{ is satisfiable} \}$$

## Variants of SAT:

(1) **Circuit-SAT**: Is a given boolean circuit $C(x_1, \dots, x_n)$ satisfiable?

**Bool. circuit:**

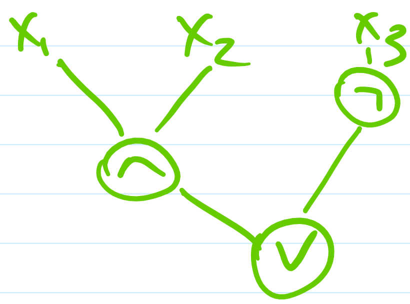



← inputs

internal gates

← output gate

SAT is a special case of Circuit-SAT: Formulas are tree-like circuits.




**Thm:** Circuit-SAT is NP-complete.

Proof: (1) $\in NP$ ✓

**Proof:** (1) $\in NP$. ✓

(2) SAT $\le_p$ Circuit-SAT. $\square$

(2) CNF-SAT: Given a CNF formula $\varphi(x_1, \ldots, x_n)$, is it satisfiable?

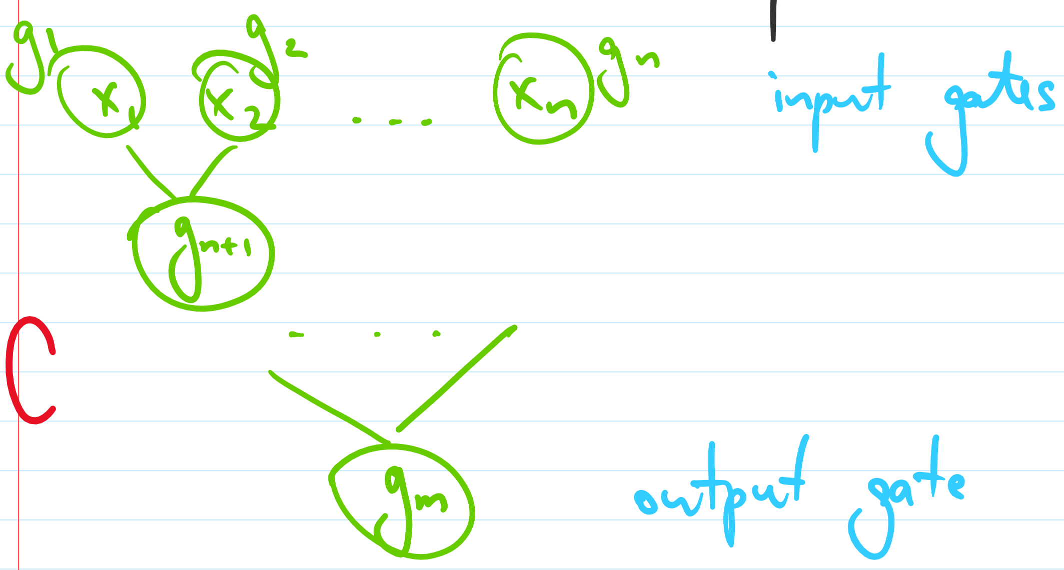
CNF = Conjunctive Normal Form
(AND of ORs of literals)

e.g. $(x_1 \vee \bar{x}_2) \wedge (\bar{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$

**Thm:** CNF-SAT is NP-complete.

**Proof:** (1) $\in NP$. ✓

(2) Circuit-SAT $\le_p$ CNF-SAT.



Introduce new variables $y_1, y_2, \dots, y_m$.

$y_i$ corresponds to gate $g_i$ of $C$

$$y_1 \equiv x_1 \ \wedge\ y_2 \equiv x_2 \ \wedge \dots \wedge\ y_n \equiv x_n$$

$$\wedge\ \text{gate}_{n+1} \ \wedge \ \dots \ \wedge\ \text{gate}_m \ \wedge\ y_m$$

Say $g_j$ : $g_k \to \wedge \leftarrow g_l$

Then define

$$\text{gate}_j : \underbrace{y_j \equiv (y_k \wedge y_l)}$$

$$\big[\, y_j \to (y_k \wedge y_l) \,\big] \ \wedge$$

$$\big[\, y_j \leftarrow (y_k \wedge y_l) \,\big]$$

$$A \to B \ \equiv\ \neg A \vee B$$

$$\neg y_j \vee (y_k \wedge y_l)$$

$\longrightarrow \neg y_j \vee (y_k \wedge y_l)$

$\equiv (\cancel{\neg y_j}^{=F} \vee y_k^{\,V2}) \wedge (\cancel{\neg y_j}^{=F} \vee y_l^{\,V2})$

$\longrightarrow \neg(y_k \wedge y_l) \vee y_j$

$\equiv (\neg y_k^{\,=T} \vee \neg y_l^{\,=T} \vee y_j^{\,=T})$

(3)

**Cor:** 3-SAT is NP-complete.

3-SAT = CNF_SAT where each clause is of width $\leq 3$

## (4) NAE-3SAT:

Given 3-CNF $\varphi(x_1, \ldots, x_n)$
$= C_1 \wedge C_2 \wedge \ldots \wedge C_m$
(clauses of width $\leq 3$)

$C_1 = x_1 \vee \overline{x_7} \vee x_{13}$

Is there a satisfying ass to $\varphi$ s.t. every clause $C_i$
... false literal ?

s.t. every clause $C_i$

has $\geq 1$ false literal ?

Observation: NAE-satisfying ass't

$a = (a_1, \dots, a_n)$

$\Rightarrow$ $\overline{a} = (\overline{a_1}, \dots, \overline{a_n})$ is also

NAE-sat. ass't.

Thm: NAE-3SAT is NP-complete.

Pf.: (1) $\in NP$ ✓

(2) Circuit-SAT $\leq_p$ NAE-3SAT

similar to our reduction

Circuit-SAT $\leq_p$ 3-SAT

clauses of length 2 + new dummy var $z$

<u>Claim:</u> orig formula (w/o z's) is SAT $\Longleftrightarrow$ $\Leftarrow$ new formula (with z's) is NAE-SAT

$\Rightarrow$