Lecture 23

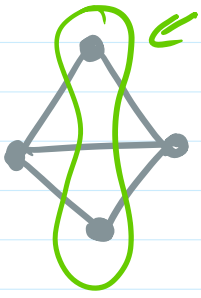
Wednesday, July 17, 2019 11:10 AM

Last time: SAT, Circuit-SAT, 3-SAT, and NAE-3SAT are NP-complete.

We'll show a few more problems NP-complete via reductions from these SAT variants.

(1) **Independent Set (IS):** Given a graph $G=(V,E)$, and an integer $K \geq 0$, decide if G has an independent set of size $\geq K$.

Ex.



← indep. set of size 2

An independent set in $G=(V,E)$ is $S \subseteq V$ s.t. $\forall u,v \in S$, $(u,v) \notin E$.

Thm: IS is NP-complete.

Proof:

(1) IS $\in$ NP. ✓

(2) 3SAT $\leq_p$ IS:

3-CNF $\varphi(x_1, x_2, \ldots, x_n) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

↓

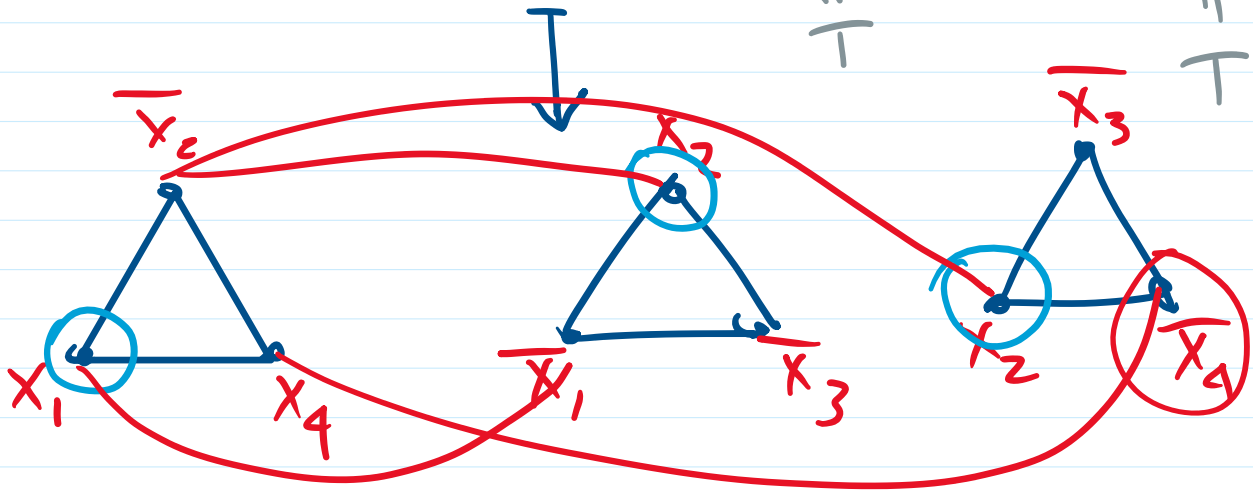
graph $G=(V,E)$ $|V| = 3 \cdot m$ $K = m$

graph $G=(V,E)$, $|V| = 3 \cdot m$, $k = m$

s.t. $\varphi \in 3SAT \iff (G, k) \in IS$

Ex. $\varphi = (x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4)$

$G =$



$\Rightarrow$ Suppose φ is satisfiable by $a = (1,1,0,0)$.
Then each clause has a true literal.
Choose exactly one true literal per clause.

Claim: The resulting set of m vertices is an independent set of G.

$\Leftarrow$ Conversely, suppose $S \subseteq V$ is an indep. set of G of size $|S| \geq m$.
Then S must contain exactly one vertex from each triangle.
Set all the corresponding literals to TRUE

from each triangle.
Set all the corresponding literals to TRUE.
(1) this is a legal truth assignment
(no inconsistencies)
(2) this assignment is satisfying.

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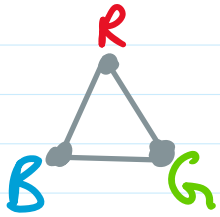
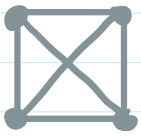
(2) 3-COL : Given a graph $G=(V,E)$, decide if G is 3-colourable.

> G is 3-colourable if there is a colouring $c : V \to \{R, G, B\}$ s.t.
> $\forall\ (u,v) \in E$, $c(u) \neq c(v)$.

Ex:





not 3-colourable

Thm: 3-COL is NP-complete.

Proof: (1) 3-COL $\in$ NP. ✓

(2) NAE-3SAT $\leq_p$ 3-COL :

3-cnf $\varphi(x_1, \ldots, x_n) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

3-cnf $\varphi(x_1, \dots, x_n) = C_1 \wedge C_2 \wedge \dots \wedge C_m$

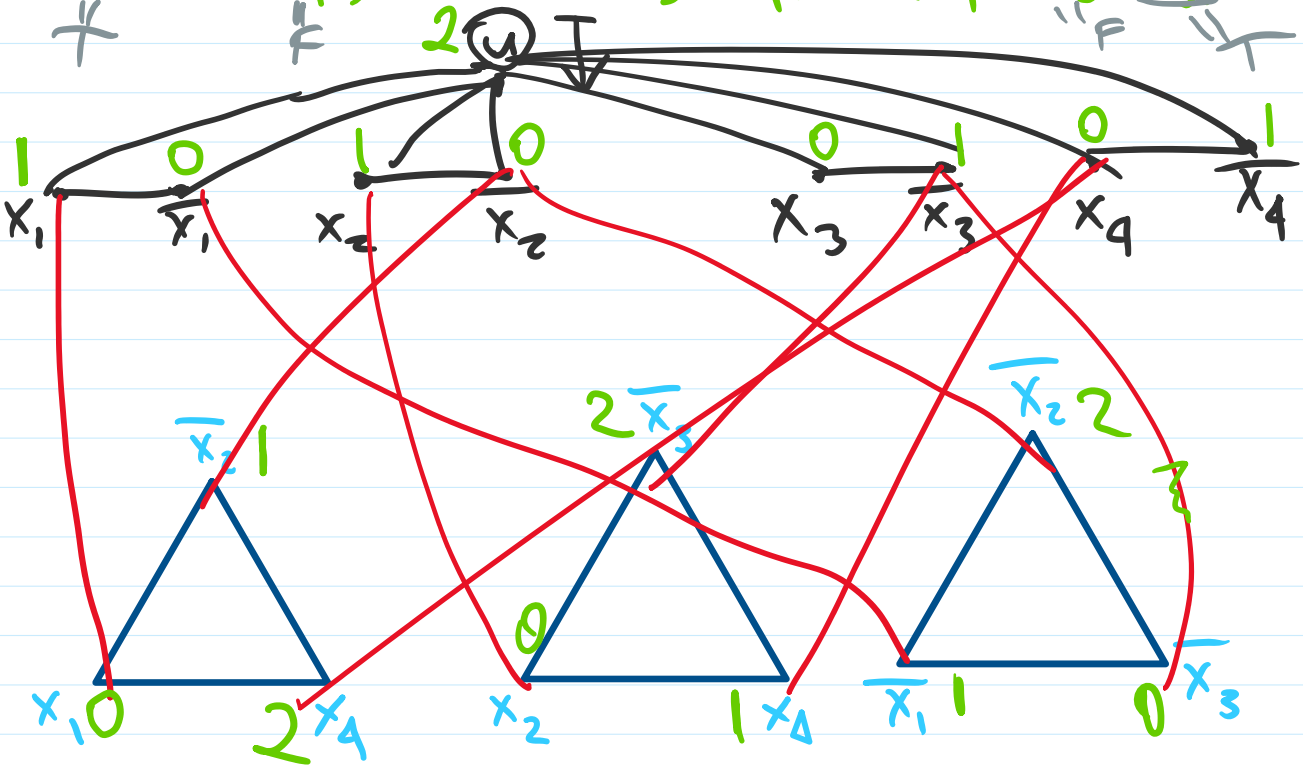
↓

graph $G = (V, E)$, $|V| = 3m + 2n + 1$

s.t. $\varphi \in$ NAE-3SAT $\Leftrightarrow$ $G \in$ 3-COL.

Ex:

$\varphi = (x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_2 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$



G: