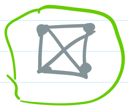
Last time: IS, 3-COLOR are NP-complete.

Clique $= \{ \langle G, k \rangle \mid$ graph G has a clique of size $\geq k \}$

a clique = subset of vertices s.t. every pair is connected by an edge

e.g. 

Thm: Clique is NP-complete.

Proof: (1) Clique $\in NP$. ✓

(2) $IS \leq_p$ Clique

$G, k \longmapsto G', k'$

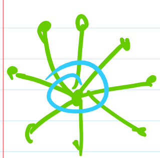
s.t. $(G, k) \in IS \iff (G', k') \in$ Clique

$IS = \overline{\text{clique}} \ldots$

Define the reduction: $(G, k) \mapsto (G^c, k)$

the complement of G

Vertex Cover (VC) $= \{ \langle G, k \rangle \mid G = (V, E)$ has a vertex cover of size $\leq k \}$



A vertex cover is $S \subseteq V$ s.t. $\forall\, e = (u, v) \in E$ ($u \in S$ OR $v \in S$)

Thm: VC is NP-complete.

Proof: (1) $VC \in NP$ ✓

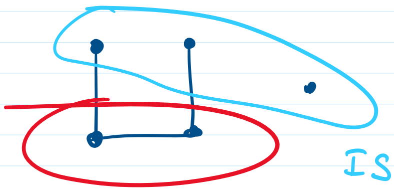
(2) $IS \leq VC$

Proof: (1) VC $\in$ NP ✓

(2) $IS \leq_p VC$

$(G,k) \longmapsto (G', k')$ s.t.

$(G,k) \in IS \iff (G',k') \in VC$



Claim: S is IS of G $\iff$ $\bar{S}$ is VC of G

Proof: $\Rightarrow$ Let S be an indep. set of G.
$\forall\, e=(u,v) \in E$, if $u \notin \bar{S}$ & $v \notin \bar{S}$, then S is not indep. set.

$\Leftarrow$ Let $\bar{S}$ be a vertex cover of G.
$\forall\, u,v \in S$, if $(u,v) \in E$, then $\bar{S}$ is not a vertex cover.

$\square$

So the $IS \leq_p VC$ reduction is:

$$(G, k) \longmapsto (G, n-k)$$

where $n = |V|$ = # nodes in G

st-HamPath = $\{ \langle G, s, t \rangle \mid G$ has a Ham. path from s to $t \}$

A Hamiltonian path is a simple path that visits every node of G.

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HamCycle $= \{ \langle G \rangle \mid G$ has a Ham. cycle$\}$

A Ham. cycle is a cycle that passes through every node of G exactly once.

Thm: st-HamPath & HamCycle are NP-complete.

Traveling Salesman Problem (TSP)

Given a complete graph G on n nodes, positive integer weights on edges $w: E \to \mathbb{Z}^+$, and an integer $W > 0$,

decide if there is a Ham cycle in G of total edge weight $\leq W$.

the sum of edge weights on the cycle

TSP is NP-complete.

(HamCycle $\leq_p$ TSP . Exercise.)

Subset Sum

Given: $a_1, a_2, \ldots, a_n, T \in \mathbb{N}$

Decide: Is there a subset $S \subseteq \{1, \ldots, n\}$ s.t.

$$\sum_{i \in S} a_i = T \quad ?$$

Thm: Subset Sum is NP-complete.

Proof: (1) Subset Sum $\in$ NP

(2) $3SAT \leq_p$ Subset Sum

(see the lecture notes for the details.) $\Box$

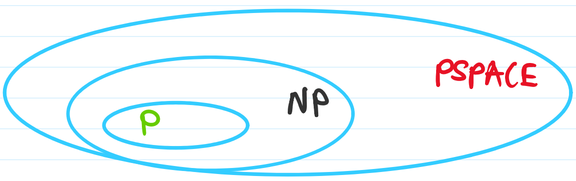
PSPACE

$$\text{PSPACE} = \bigcup_{c \geq 0} \text{SPACE}(n^c)$$

space: # tape cells touched by a TM during its run

$$\text{NPSPACE} = \bigcup_{c \geq 0} \text{NSPACE}(n^c)$$

max space $\leq n^c$ over all nondeterministic computation branches of a given NTM.



Thm: NP $\subseteq$ PSPACE. (Exercise!)

Thm [Savitch's Theorem]:

NPSPACE = PSPACE.

More precisely,

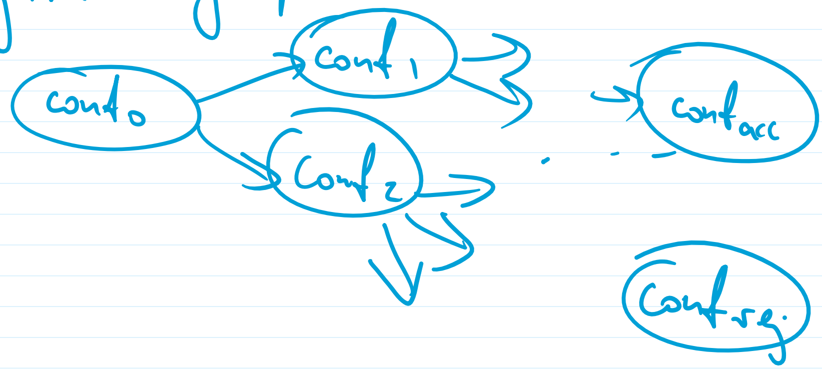
$$\text{NSPACE}(s) \subseteq \text{SPACE}(s^2)$$

Proof:

NTM M, x, $|x| = n$

uses space $\leq S = S(n)$.

Configuration graph:



conf$_i$:

$O(1)$ bits (1) state of M

$O(\log_2 n)$ bits (2) position on input tape

$O(S)$ bits (3) contents of all worktapes

$O(\log_2 S)$ bits (4) positions on the worktapes

Σ: $O(S + \log_2 n)$ bits $= O(S)$

(assume $\underline{S \geq \log_2 n}$)

\# configs $\leq 2^{O(S)}$

$\shortparallel$

Nodes of Config Graph

conf $\longrightarrow$ conf'

iff

M in conf goes to conf' (legal move)

Our task: , $O(S)$.

Our task:

digraph G on $N = 2^{O(s)}$ nodes

$s, t \in$ nodes of G

init config (for s), accepting config (for t)

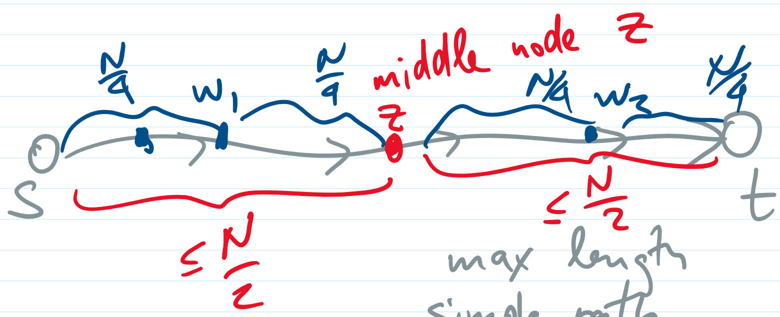
Decide: Is there $s \rightsquigarrow t$ in G?

BFS: will solve the reachability problem.

time: linear in the size of the input graph

In our case, time $2^{O(s)}$ time.

Space is also $\geq 2^s$. Too much!



max length simple path is on $\leq N$ edges

Main call: Path(init, accept, $O(s)$)

algo Path (x, y, i)

% check if $x \xrightarrow{\leq 2^i \text{ steps}} y$

if $i = 0$ **then**

if $i = 0$ then
  Accept if $(x = y$ or $(x,y) \in E)$
endif

for every node $z \in V$
  if $\text{Path}(x, z, i-1)$ AND $\text{Path}(z, y, i-1)$
  then Accept
  endif
endfor
Reject

re-use space

$$\text{Space}(i) = \text{Space}(i-1) + O(|x|)$$

$$\leq O(|x|^2)$$

N nodes $\quad |x| \leq \log N$

$$\text{Space}(\log N) \leq O((\log N)^2)$$

$$\text{Space}(m) = \text{Space}(m-1) + a$$

$$\Downarrow$$

$$\text{Space}(m) = O(m \cdot a)$$