Last time: Savitch's Theorem:

$$NSPACE(s) \subseteq SPACE(s^2)$$

(for "nice" functions $s: \mathbb{N} \to \mathbb{N}$).

Hence, $NPSPACE = PSPACE$.

PSPACE-complete problem

Quantified Boolean Formulas:

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots Q_n x_n \; \underbrace{\varphi(x_1, x_2, \ldots, x_n)}_{\text{CNF formula}}$$

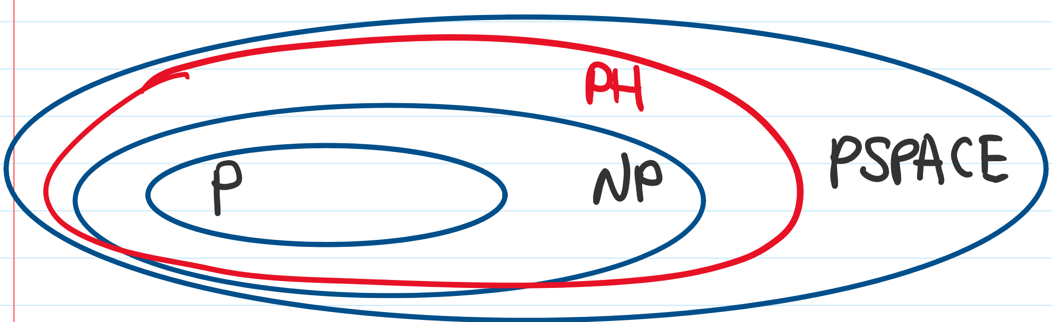
bin. string

alternating quantifiers

Given: A QBF $\exists x_1 \forall x_2 \ldots Q x_n \varphi(x_1, x_2, \ldots, x_n)$.

Decide: Is this QBF true?

Recall:



NP-complete problem: SAT

$\exists x \;\; \varphi(x)$

(special case of QBF: one quantifier only)

co-NP-complete: $\overline{\text{SAT}}$ (TAUT)

$\forall x \;\; \varphi(x)$

[NP vs. coNP
proof complexity]

$\underbrace{\underline{\forall} x_1 \, \exists x_2}_{\text{2 quantifiers}} \; \underline{\varphi(x_1, x_2)}$ Π_2^p

$\underline{\exists} x_1 \, \forall x_2 \ldots Q_k x_k \; \underline{\varphi(x_1, \ldots, x_k)}$

k is a constant Σ_k^p

Polynomial-Time Hierarchy

$$PH = \bigcup_{i \geq 0} \Sigma_i^p$$

$$\Sigma_0^p = P, \quad \Sigma_1^p = NP, \Sigma_2, \ldots$$

$$\Pi_1^p = coNP, \Pi_2, \ldots$$

Open: Is PH infinite ?

TQBF = { qbf ψ | ψ is true }

True QBF

Thm: TQBF is PSPACE-complete.

Proof: (1) TQBF $\in$ PSPACE. ✓

(2) $\forall L \in$ PSPACE, $\Leftarrow$

$$L \leq_p \text{TQBF}.$$

ψ =

(1) $\exists x_1 \forall x_2 \exists x_3 \ldots Q x_n \; \varphi(x_1, \ldots, x_n)$

Algo Truth (ψ)

if ψ has no quantifiers

then return Value (ψ)

endif

for all a_1,

Truth ($\forall x_2 \ldots Q x_n \; \varphi(a_1, x_2, \ldots, x_n)$)

Truth $\left(\forall x_2 \ldots Q x_n \, \varphi(a_1, x_2, \ldots, x_n)\right)$

<u>if</u> find any a_1 s.t. this is true

<u>then</u> return True

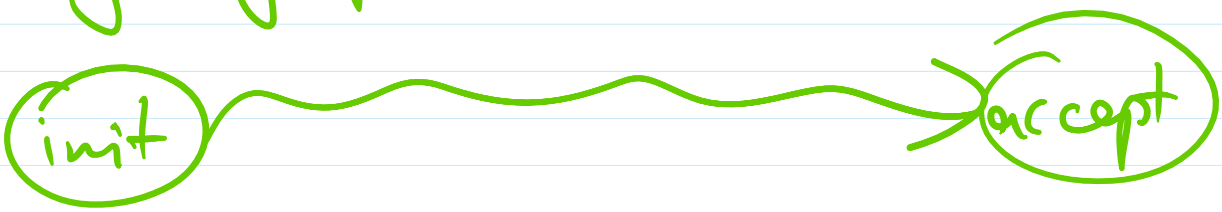
<u>else</u> return False.

$L \in \mathrm{Space}(\underline{\underline{S}})$, $S = \mathrm{poly}_S(n)$

$\#$ nodes in config graph ≈ 2

$\underline{x}$, $|x| = n$, $\underline{x} \in L$?

config. graph:



<u>Want:</u> $\psi(X,Y) = \begin{cases} \text{True} & \text{if } X \rightsquigarrow Y \\ \text{False} & \text{o.w.} \end{cases}$

$\psi_0(X,Y) = \begin{cases} \text{True} & \text{if } X \xrightarrow{\le 1 = 2^0} Y \\ \text{False} & \text{o.w.} \end{cases}$

$\psi_i(X,Y) = \begin{cases} \text{True} & \text{if } X \xrightarrow{\le 2^i} Y \\ \text{False} & \text{o.w.} \end{cases}$

Need: $\psi_m(\text{init}, \text{accept})$

$$m = \log |V| = \log |2^s| = s = \text{poly}(n)$$

$$\psi_i \longmapsto \psi_{i+1}$$

$$X \xrightarrow{\le 2^i} Y \qquad X \xrightarrow{\le 2^{i+1}} Y$$

$$X \xrightarrow{\le 2^i} Z \xrightarrow{\le 2^i} Y$$

$$\psi_{i+1}(X, Y) \equiv \exists Z \; \psi_i(X, Z) \wedge \psi_i(Z, Y)$$

$$|\psi_{i+1}| \geq 2 \cdot |\psi_i|$$

$$\psi_{i+1}(X,Y) \equiv$$

$$\exists Z \,\forall A \,\forall B$$

$$\Big(\big((A,B)=(X,Z) \vee (A,B)=(Z,Y) \big) \Rightarrow \psi_i(A,B) \Big)$$

$$|\psi_{i+1}| \le |\psi_i| + \mathrm{poly}(n)$$