1. Describe two different Turing machines, $M$ and $N$, such that, when started on any inputs, $M$ outputs $\langle N \rangle$ and $N$ outputs $\langle M \rangle$.

2. Using the Recursion Theorem, complete the following, alternative proof that $A_{TM}$ is undecidable.

   Suppose $A_{TM}$ is decidable by a decider TM $H$. Define a new TM $M$ = “On input $w$, get own description $\langle M \rangle$; simulate $H$ on $\langle M, w \rangle$; if $H$ accepts, then reject; if $H$ rejects, then accept.”

   Derive a contradiction by analyzing the question: Does $M$ accept the empty string $\epsilon$?

3. **Kolmogorov complexity**

   (a) Define $L = \{x \mid K(x) \geq |x|\}$, where $K(x)$ is the Kolmogorov complexity of the binary string $x$. Prove that $L$ is undecidable.

   (b) Show that the set $\{x \mid K(x) \geq |x|\}$ of incompressible strings contains no infinite subset that is semi-decidable.

4. For each $m > 1$ let $Z_m = \{0, 1, \ldots, m - 1\}$. Consider arithmetic formulas over $Z_m$ where addition and multiplication operations are interpreted as addition modulo $m$ and multiplication modulo $m$, respectively, and where the variables are assumed to take values from $Z_m$. Argue that, for each $m > 1$, the language of true arithmetic sentences over $Z_m$ is decidable. That is, argue that for each fixed $m > 1$, there is an algorithm for deciding if a given arithmetic sentence over $Z_m$ is true or false.

5. Recall that a proof in a proof system $P$ is a sequence of formulas such that each formula is either an axiom of $P$ or follows from some earlier formulas in the sequence by inference rules of $P$. Suppose that a proof system $P$ has finitely many inference rules, but infinitely many axioms. However, the axioms are enumerable by a TM $A$: the TM $A$ when run on the empty string $\epsilon$ will be outputting axioms $a_i$ so that every axiom of $P$ is eventually output. Define the language

   \[ \text{Provable} = \{\langle \phi \rangle \mid P \text{ proves } \phi \}. \]

   Argue that the language Provable is semi-decidable.