Asset Prices with Heterogeneity in Preferences and Beliefs

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In this paper, we study asset prices in a dynamic, continuous-time, and general-equilibrium endowment economy in which agents have "catching up with the Joneses" utility functions and differ with respect to their beliefs (because of differences in priors) and their preference parameters for time discount, risk aversion, and sensitivity to habit. A key contribution of our paper is to demonstrate how one can obtain a closed-form solution to the consumption-sharing rule for agents who have both heterogeneous priors and heterogeneous preferences without restricting the risk aversion of the two agents to special values. We solve in closed form also for the state-price density, the risk-free interest rate and market price of risk, the stock price, equity risk premium, and volatility of stock returns, the term structure of interest rates, and the conditions necessary to obtain a stationary equilibrium in which both agents survive in the long run. The methodology we develop is sufficiently general in that, as long as markets are complete, it can be used to obtain the sharing rule and state prices for models set in discrete or continuous time and for arbitrary endowment and belief updating processes. (*JEL* G12, D51, D53, D91)

Two key characteristics of economic agents are their beliefs and preferences. Our objective in this paper is to study the effect of heterogeneity in both of these characteristics on the consumption and portfolio choices of individual agents, and the resulting asset prices. The agents we study have different beliefs about the growth rate of the aggregate endowment process and "catching up with the

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Joneses" utility functions, with different parameters for patience, risk aversion, and sensitivity to the historical standard of living. We show how to solve *in closed form* for optimal policies and asset prices of the stock and bond in a general equilibrium stochastic dynamic exchange economy with heterogeneous agents. This allows us to identify the strengths and limitations of the model with heterogeneity in both preferences and beliefs.

The importance of studying models with heterogeneous agents rather than a representative agent has been recognized by both policy makers and academics. For instance, in his Ely lecture. Hansen (2007) says: "While introducing heterogeneity among investors will complicate model solution, it has intriguing possibilities. … There is much more to be done." Similarly, in his presidential address to the American Economic Association, Sargent (2008) discusses extensively the implications of the common beliefs assumption for policy. Empirical work by Beber, Buraschi, and Breedon (2010), Berrada and Hugonnier (2011), Buraschi and Jiltsov (2006), Buraschi, Trojani, and Vedolin (2009, 2010), and Ziegler (2007) also suggest the importance of allowing for heterogeneous beliefs and preferences in models of asset pricing.

A key contribution of our paper is to demonstrate how one can obtain a closed-form solution to the consumption-sharing rule for agents who have both heterogeneous priors and heterogeneous preferences without restricting the risk aversion of the two agents to special values.[1] In the case of two agents, the consumption-sharing rule is a nonlinear algebraic equation that reduces to a polynomial of degree η if the ratio of the risk aversion of one agent to that of the other is a natural number. If η equals two, three, or four, then this polynomial equation of course can be solved in closed form. We show how to construct a closed-form solution for all real values of η, using a theorem attributed to Lagrange, and we solve in closed form for the interest rate and market price of risk and express the stock price, equity premium, volatility of stock market returns, and the term structure of interest rates as deterministic one-dimensional integrals, when agents have heterogeneous preferences and beliefs. Thus, the model we analyze nests the models that consider an exchange economy with agents who have expected utility with different degrees of risk aversion, such as Wang (1996), Bhamra and Uppal (2009), and Weinbaum (2012), models with "catching up with the Joneses" utility functions, as in Chan and Kogan (2002) and Xiouros and Zapatero (2010), and models in which agents have expected utility with heterogeneous beliefs, for instance, Basak (2005) and Yan (2008).

The methodology we develop is sufficiently general in that, as long as markets are complete, it can be used to obtain the sharing rule and state prices for models set in discrete or continuous time and for arbitrary endowment and

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[1] Our work can be viewed as complementary to that of Calin et al. (2005), who provide an analytic representation (that is, a convergent power series) for the price-dividend function of one state variable in an economy with a *single* representative agent whose utility function displays habit formation, and to Garlappi and Skoulakis (2011), who show how to exploit Taylor series expansions to solve portfolio choice problems in partial equilibrium.

belief updating processes. We consider the "catching up with the Joneses" utility function that has external habit, but in contrast with Chan and Kogan (2002) and Xiouros and Zapatero (2010), we allow the sensitivity to the historical standard of living to be agent specific. Our specification nests isoelastic and logarithmic utility functions, and is straightforward to apply to other time-additive utility functions, such as exponential and quadratic utility. Given the ubiquity of nonlinear sharing rules in solutions to problems in economics, finance, and decision theory (see Peluso and Trannoy (2007) for examples of such problems), the approach we develop also can be applied to other problems that previously would have called for numerical methods.

The paper that is closest to our work is Cvitanić et al. (2012), who also study asset prices in an economy in which agents have expected utility and differ with respect to both beliefs and their preference parameters. Their paper provides *bounds* on asset prices and characterizes prices *in the limit* when only one agent survives. However, it does not provide closed-form solutions for these quantities. In contrast with Cvitanić et al. (2012), we provide a closed-form solution for the stochastic discount factor without restricting the risk aversion of the two agents to special values and also allowing for learning. In particular, we show how the stochastic discount factor can be expressed as a weighted average of stochastic discount factors from a set of underlying *single-agent* economies.[2]

Most of the other papers in the existing literature with heterogeneous agents allow for either differences in beliefs or differences in preferences. We first discuss the literature that considers heterogeneity in beliefs and then the literature that considers differences in preferences. Essentially, there are two ways to generate heterogeneity in beliefs. In the first approach, agents receive different information. This is the classical approach, adopted in the early noisy-rational-expectations literature with asymmetric information.[3] In this class of models, one group of (informed) agents receives private signals, and then there is a second group of agents (noise-traders) that trades for exogenous reasons and thereby prevents the price from fully revealing the private information of the informed agents. The second approach for generating heterogeneity, which is the one we adopt, is to have agents who "agree to disagree" about some aspect of the underlying economy, and in this class of models, it is assumed that agents do not learn from each other's behavior. Morris (1995) provides a

[2] We should point out that, in contrast to our analysis, which is for the case of two agents, the limit analysis of Cvitanić et al. (2012) considers an economy with more than two agents and derives interesting implications for the term structure of interest rates. Muraviev (2013) extends the analysis in Cvitanić et al. (2012) to the case with learning and in which agents have "catching up with the Joneses" utility functions considered in Chan and Kogan (2002) but in which the sensitivity to the historical standard of living is agent specific, whereas Borovička (2012) extends the analysis to the case of recursive preferences.

[3] See, for instance, Grossman and Stiglitz (1980), Hellwig (1980), Wang (1993), and Shefrin and Statman (1994).

good philosophical discussion of this modeling approach.[4] Excellent reviews of this literature are provided by Basak (2005) and Jouini and Napp (2007).

We now discuss the literature on the effect of heterogeneous preferences on asset prices. The effect of different time discount factors on the efficient allocation of consumption is studied by Gollier and Zeckhauser (2005). The effect of heterogeneity in risk aversion on asset prices is examined in several papers, most of which assume that investors have expected utility.[5] In contrast with the papers that assume investors have expected utility, Chan and Kogan (2002) and Xiouros and Zapatero (2010) study asset prices in an economy in which agents have "catching up with the Joneses" preferences, where habit formation ensures that the model is stationary. Finally, there are papers that work with Epstein and Zin (1989) recursive preferences.[6]

When there are multiple agents who differ in their risk aversion, there is no paper in the literature that provides a complete characterization of equilibrium that is exact and entirely analytical. For example, for the case of expected utility, Wang (1996) provides closed-form expressions for only particular parameter values; Kogan and Uppal (2001) characterize the equilibrium in production and exchange economies approximately using perturbation analysis in the neighborhood of log utility; Bhamra and Uppal (2009) and Tran (2009) study stock-market-return volatility, but solve numerically for volatility; Dumas (1989) solves numerically for the interest rate in a production economy; for the case of "catching up with the Joneses" preferences, Chan and Kogan (2002) rely on numerical solutions; and Xiouros and Zapatero (2010) provide an expression for the value function of the central planner assuming a Gamma distribution for the risk tolerances of the investors, but asset prices are obtained using numerical methods.

[4] Examples of papers using such models of incomplete information include Basak (2000), Beber, Buraschi, and Breedon (2010), Berrada (2006), Borovička (2012), Buraschi and Jiltsov (2006), Buraschi, Trojani, and Vedolin (2009, 2010), Cecchetti et al. (2000), David (2008), David and Veronesi (2002), Duffie, Garleânu, and Pedersen (2002), Dumas, Kurshev, and Uppal (2009), Gallmeyer (2000), Gallmeyer and Hollifield (2008), Kogan et al. (2006), Scheinkman and Xiong (2003), Veronesi (1999), Xiong and Yan (2010), Yan (2008), and Zapatero (1998). Yan (2008) also studies a model in which agents have both heterogeneous beliefs and preferences, but he solves for asset prices in terms of exogenous variables only for the case in which both agents have the same risk aversion, which is a natural number (see his Proposition 3).

[5] For example, Dumas (1989) studies the risk-free rate and the risk premium in a production economy; Wang (1996) examines the term structure in an exchange economy; Basak and Cuoco (1998), Kogan, Makarov, and Uppal (2007), and Chabakauri (2013) analyze the effect of constraints on borrowing and short-sales on the equity risk premium in an exchange economy; Bhamra and Uppal (2009) and Tran (2009) examine the volatility of stock market returns; Benninga and Mayshar (2000) and Weinbaum (2009) study option prices; Longstaff and Wang (2012) investigate the relation between open interest in the bond market and stock market returns; Cvitanić and Malamud (2009a, 2009b, 2009c) consider equilibrium with multiple heterogeneous traders who maximize utility of only terminal wealth; and, Garleânu and Panageas (2008) study the effect of heterogeneous preferences in an overlapping generations model that leads to a stationary equilibrium.

[6] For example, Guvenen (2009), studies asset pricing in a model with heterogeneity in elasticity of intertemporal substitutio; Isaenko (2008) studies the term structure in a model in which agents differ in both their risk aversion and elasticity of intertemporal substitution; and Gomes and Michaelides (2008) study portfolio decisions of households and asset prices in a model in which agents are heterogeneous not just in terms of preferences but are also exposed to uninsurable income shocks in the presence of borrowing constraints.

The rest of this paper is arranged as follows. In Section 1, we describe our model of an exchange economy with heterogeneous agents. The equilibrium consumption-sharing rule, derived by solving the problem of a "central planner," is given in Section 2 along with the state price density. Section 3 gives the dynamics of the state price density, defined in terms of the risk-free rate and the market price of risk. Asset prices, including the stock price, the volatility of stock market returns, and the equity and term premium, are given in Section 4. In Section 5, we show how to price financial assets when the logarithm of aggregate endowment and agents' beliefs follow general affine processes, instead of the geometric Brownian motion assumed for the endowment process and the exponential martingale process assumed for beliefs in the previous section. We conclude in Section 6. Our main results are highlighted in propositions; results for special cases are given in corollaries; and detailed proofs for all the results, including a statement of Lagrange's theorem, are provided in the Appendix.

1. The Model

In this section, we describe the features of the economy being considered. We consider a continuous-time, pure-exchange, Arrow-Debreu (complete markets) economy with an infinite time horizon. There is a single nonstorable consumption good that serves as the numeraire and is modeled as an exogenously specified stochastic endowment process, Y_t, that is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

There are a large number of investors in the economy. These investors are of two types, which we denote by $k \in \{1,2\}$. We adopt the convention of subscripting by k the quantities related to agents of type k, where $k \in \{1,2\}$. To be concise, instead of referring to these investors as "agents of type 1 and 2," we simply call them Agent 1 and Agent 2.

The utility function of Agent k is denoted by U_k, and the beliefs of Agent k are denoted by the probability measure $\mathbb{P}^k$. Our model differs from the standard representative-agent Lucas (1978) model along two dimensions: first, preferences of the two agents are heterogeneous; second, the two agents may not have the correct beliefs about the aggregate endowment process, and the beliefs of one agent may differ from those of the other. The sharing rule of the investor and the equilibrium state price density do *not* depend on the particular processes chosen for the dynamics of endowments and beliefs.[7] The only assumption we need for the results regarding static quantities, such as the sharing rule and the state price density, is that financial markets are dynamically

[7] For instance, the endowment process could be a geometric Brownian motion that is typically assumed, but it could also be a much more complicated process that is not necessarily continuous or even affine. Similarly, whereas it is possible to assume that beliefs are not updated at all, one could also assume Bayesian updating or some form of non-Bayesian updating.

complete. Therefore, we specify the processes for endowments and beliefs only in Section 3, where we study the dynamics of the consumption share, state price density, and asset returns.

1.1 Preferences of the two agents

The consumption of Agent k at instant u is denoted by $C_{k,u}$ and the instantaneous utility from consumption is given by the following power function that depends on consumption relative to habit, $H_{k,u}$:

$$U_k(C_{k,u}, H_{k,u}) \equiv e^{-\beta_k u} \frac{1}{1-\gamma_k} \left(\frac{C_{k,u}}{H_{k,u}} \right)^{1-\gamma_k}, \tag{1}$$

where β_k is the constant subjective discount rate (that is, the rate of time preference), and γ_k is the degree of relative-risk aversion.[8] Without loss of generality, we assume that Agent 1's relative-risk aversion is less than that of Agent 2: $\gamma_1 < \gamma_2$.

The quantity $H_{k,u}$ in (1) can be interpreted as Agent k's sensitivity to the historical standard of living (external habit), as modeled in Muraviev (2013), which generalizes the specification in Chan and Kogan (2002) to allow for agent-specific habit. Under the approach in Muraviev,

$$H_{k,u} = X_u^{h_k} = e^{h_k x_u}, \text{ for } h_k > 0,$$

and $x_u \equiv \log X_u$. One can then define x_t as the weighted geometric average of past realizations of the logarithm of the aggregate endowment process:

$$x_t = x_0 e^{-\lambda_x t} + \lambda_x \int_0^t e^{-\lambda_x (t-u)} y_u \, du, \tag{2}$$

where $y_u \equiv \log Y_u$, and Y_u denotes the aggregate endowment at time u. So, whereas X_u represents the general index for the standard of living, the scalar h_k determines the sensitivity of Agent k to this index; if $h_k = 1$ for all k, this reduces to the specification in Chan and Kogan (2002), and if $h_k = 0$, one gets the standard isoelastic utility function without habit.

We define the distance between the logarithm of the aggregate endowment and its weighted geometric average as

$$\omega_t \equiv y_t - x_t. \tag{3}$$

Observe that (2) implies that $dx_t = \lambda_x (y_t - x_t) dt$, and so the evolution of ω_t is given by

$$d\omega_t = \lambda_x \left(\lambda_x^{-1} E_t \left[\frac{dy_t}{dt} \right] - \omega_t \right) dt + dy_t - E_t[dy_t], \tag{4}$$

which makes clear that ω_t exhibits mean reversion for any specification of the aggregate endowment.

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[8] See Chan and Kogan (2002) for a discussion of this specification for the utility function, and why it is still appropriate to interpret γ as the coefficient of relative-risk aversion. For other papers in the literature that study the effect of habit on asset prices in representative-agent models, see Abel (1990, 1999) and Constantinides (1990).

Chan and Kogan (2002) explain that γ_k represents the relative-risk aversion of Agent k. We explain below that if we parameterize h_k as in (5) below

$$h_k \equiv \frac{\gamma_k - \frac{1}{\psi_k}}{\gamma_k - 1}, \tag{5}$$

then we can interpret $\frac{1}{\psi_k}$ as the sensitivity of the risk-free rate to the growth rate of aggregate consumption in the steady state when Agent k is the sole agent in the economy without risk.[9]

1.2 The optimization problem of each agent

Given her beliefs, represented by the probability measure $\mathbb{P}^k$, the expected lifetime utility of Agent k at time t from consuming $C_{k,u}$ is given by

$$V_{k,t} = E_t^k\left[\int_t^{\infty} e^{-\beta_k(u-t)} \frac{1}{1-\gamma_k}\left(\frac{C_{k,u}}{H_{k,u}}\right)^{1-\gamma_k} du\right], \tag{6}$$

where E_t^k denotes the time-t conditional expectation operator with respect to the probability measure $\mathbb{P}^k$.

The problem of Agent k is to maximize lifetime utility, given by $V_{k,0}$ in (6), which is subject to a static budget constraint that restricts the present value of all future consumption to be no more than the initial wealth of each agent, denoted by $W_{k,0}$:[10]

$$E_0^k\left[\int_0^{\infty} \frac{\pi_{k,t}}{\pi_{k,0}} C_{k,t}\, dt\right] \leq W_{k,0}, \tag{7}$$

9 We start by considering the marginal utility (MU) of consumption at date t for Agent k, which is given by

$$\text{MU}_{k,t} = e^{-\beta_k t} H_{k,t}{}^{\gamma_k - 1} C_{k,t}^{-\gamma_k}.$$

When Agent k is the sole agent in the economy, $C_k = Y$, and so her marginal utility can be written as:

$$\text{MU}_{k,t} = e^{-\beta_k t} e^{-\left(\gamma_k - \frac{1}{\psi_k}\right)\omega_t} e^{-\frac{1}{\psi_k} y_t}.$$

Thus, the instantaneous interest rate in the deterministic version of the economy is $r_t = -\ln MU_t = \beta_k + \left(\gamma_k - \frac{1}{\psi_k}\right)\frac{d\omega_t}{dt} + \frac{1}{\psi_k}\frac{dy_t}{dt}$. From (4), we can see that in the deterministic version of the economy, ω possesses a steady state. At the steady state, $\frac{d\omega_t}{dt} = 0$, so

$$\frac{\partial r_t |_{\frac{d\omega_t}{dt}=0}}{\partial\left(\frac{dy_t}{dt}\right)} = \frac{1}{\psi_k}.$$

One might be tempted to link ψ_k with the elasticity of intertemporal substitution, but a simple calculation shows that in a model with internal habit, the elasticity of intertemporal substitution between dates t and $u > t$ is given by $\left[\frac{1}{\psi_k} + \frac{\left(\gamma_k - \frac{1}{\psi_k}\right)\left(1 - e^{-\lambda_x(u-t)}\right)}{\lambda_x(u-t)}\right]^{-1}$.

10 The budget constraint for Agent k in (7) is written in terms of the state prices perceived by this agent; one could write an equivalent expression in terms of the state prices (and expectation) of the central planner.

in which $\pi_{k,t}$ is the marginal utility of investor k at date t under the probability measure $\mathbb{P}^k$:

$$\pi_{k,t} \equiv \frac{\partial U(C_{k,t})}{\partial C_{k,t}} = e^{-\beta_k t} \left(\frac{1}{H_{k,t}} \right)^{1-\gamma_k} C_{k,t}^{-\gamma_k} = e^{-\beta_k t} e^{\left(\gamma_k - \frac{1}{\psi_k}\right)(y_t - \omega_t) - \gamma_k \ln C_{k,t}}. \quad (8)$$

1.3 The equilibrium

We use a notion of equilibrium that is an extension of equilibrium in the single-agent model of Lucas (1978): both agents optimize their expected lifetime utility and all markets clear. Given our assumption that preferences are time separable and financial markets are complete, the dynamic consumption-portfolio choice problem simplifies to a static problem that requires one to choose the optimal allocation of consumption between the two investors for each date and state. If agents have identical beliefs, then one can solve for the equilibrium consumption policies by maximizing the "central-planner's" social-welfare function, which is a weighted average of the utility functions of individual agents, subject to the resource constraint that aggregate consumption is equal to aggregate endowment (dividends). In contrast with the case of identical beliefs, Basak (2005) shows that if agents have heterogeneous beliefs, the weights used to construct the central planner's utility function are stochastic. The central planner's utility function in this case is given by

$$\sup_{C_1+C_2 \leq Y} \sum_{k=1}^{2} \lambda_{k,t} U_k(C_{k,t}), \quad (9)$$

where

$$\lambda_{k,t} = \lambda_{k,0} \xi_{k,t},$$

and $\xi_{k,t}$ is the Radon-Nikodym derivative $d\mathbb{P}^k/d\mathbb{P}$, which relates Agent k's subjective beliefs to the true physical probability measure.[11]

2. Consumption Share, State Price Density, and Asset Prices

Our main contribution is a closed-form and convergent series solutions for the sharing rule in Proposition 1. In addition, we show in Proposition 2 how to construct in two simple steps the state-price density in the heterogeneous-agent economy. In the first step, one needs to obtain only the state price density for an economy in which there is just one type of agent; this is obtained by computing the marginal utility of consumption for a single agent. In the second step, one needs to combine the state-price densities from the two single-agent economies to obtain the state price density in the heterogeneous-agent economy; we

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11 If the state space for aggregate dividends were discrete, then $\xi_{k,t}$ would be the ratio of the probability that Agent k assigns to a particular state, relative to the true probability of that state.

provide the formula for doing this. These results are derived without making specific assumptions about the process for aggregate endowment and how the beliefs of the two agents are updated.

Once we have obtained the state-price density for the heterogeneous-agent economy, the price for any asset can be obtained by integrating the product of the asset's payoff and the state-price density and taking expectations of that integral; we state this result in Proposition 3, and we illustrate it for particular processes for endowments and beliefs in Section 4 and for general affine processes in Section 5.

2.1 The equilibrium consumption-sharing rule

The consumption-sharing rule, which shows how aggregate consumption is allocated between the two agents in equilibrium, is given by the first-order condition for optimal consumption for the central planner's problem in (9):

$$(\lambda_{1,0}\xi_{1,t})e^{-\beta_1 t}\left(\frac{1}{H_{1,t}}\right)^{1-\gamma_1} C_{1,t}^{-\gamma_1} = (\lambda_{2,0}\xi_{2,t})e^{-\beta_2 t}\left(\frac{1}{H_{2,t}}\right)^{1-\gamma_2} C_{2,t}^{-\gamma_2}. \tag{10}$$

To solve explicitly for the equilibrium allocations, we write Agent k's consumption share as $\nu_{k,t} \equiv \frac{C_{k,t}}{Y_t}$, where $0 \le \nu_k \le 1$, and $\nu_1 + \nu_2 = 1$. Then Equation (10) can be written as

$$\lambda_{1,0}\xi_{1,t}\, e^{-\beta_1 t}\left(\frac{1}{H_{1,t}}\right)^{1-\gamma_1} Y_t^{-\gamma_1}\nu_{1,t}^{-\gamma_1} = \lambda_{2,0}\xi_{2,t}\, e^{-\beta_2 t}\left(\frac{1}{H_{1,t}}\right)^{1-\gamma_1} Y_t^{-\gamma_2}\nu_{2,t}^{-\gamma_2}.$$

Defining π_t to be the equilibrium state-price density, the expression above can be rewritten as

$$\pi_t = \hat{\pi}_{1,t}\,\nu_{1,t}^{-\gamma_1} = \hat{\pi}_{2,t}\,\nu_{2,t}^{-\gamma_2}, \tag{11}$$

where $\hat{\pi}_{k,t}$, defined below, is the state-price density under the physical probability measure $\mathbb{P}$ when Agent k is the sole agent in the economy:

$$\hat{\pi}_{k,t} = \lambda_{k,0}\xi_{k,t}\, e^{-\beta_k t}\left(\frac{1}{H_{k,t}}\right)^{1-\gamma_k} Y_t^{-\gamma_k} = \lambda_{k,0}\xi_{k,t}\, e^{-\beta_k t} e^{-\left(\gamma_k - \frac{1}{\psi_k}\right)\omega_t - \frac{1}{\psi_k} y_t}. \tag{12}$$

Thus, the consumption-sharing rule in (11) can be expressed as

$$\nu_{2,t}^{\eta} A_t = \nu_{1,t}, \tag{13}$$

$$\text{where} \qquad A_t = \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{1}{\gamma_1}},$$

$$\text{and} \qquad \eta = \gamma_2/\gamma_1.$$

When $\eta \in \{1,2,3,4\}$, the expression in (13) can be written as a polynomial of degree 4 or less, thus allowing us to solve for the equilibrium consumption allocation in terms of radicals, using standard results from polynomial theory, as pointed out in Wang (1996). Because polynomials of order 5 and higher do not admit closed-form solutions in terms of radicals, we are able to go beyond the results in Wang (1996) by solving for the consumption-sharing rule in closed-form when η is a natural number greater than or equal to five. However, when η is a natural number greater than or equal to five, the consumption shares can be obtained in closed form by using hypergeometric functions.[12] We go further by showing that when η is *any real number*, we are able to derive closed-form and convergent series solutions for the sharing rule. The series solutions are derived using a theorem by Lagrange (see the Appendix), which, to the best of our knowledge, has not been used before in finance or economics. However, Lagrange's theorem does not provide the radius of convergence for the series, which is essential if we want to use these series to study the behavior of the consumption shares. We show, in the proof of Proposition 1, how to identify the radius of convergence: depending on whether $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R$ or $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R$, we get a different convergent series for the sharing rule and the solutions corresponding to these two regions are given in (14), with the two series being continuous at R.

Proposition 1 (Equilibrium share of consumption). Agent 2's equilibrium share of the aggregate endowment, $\nu_{2,t} = \frac{C_{2,t}}{Y_t}$, is given by

$$\nu_{2,t} = \begin{cases} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}} \binom{n\frac{\gamma_1}{\gamma_2}}{n-1} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R, \\ 1 - \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}} \binom{n\frac{\gamma_2}{\gamma_1}}{n-1} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \end{cases} \tag{14}$$

where

$$R = \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\gamma_2 - \gamma_1} = \left(\frac{(\eta-1)^{\eta-1}}{\eta^{\eta}}\right)^{\gamma_1}, \tag{15}$$

and, for $z \in \mathbb{C}$ and $k \in \mathbb{N}$, $\binom{z}{k} = \Pi_{j=1}^{k} \frac{z-k+j}{j}$ is the generalized binomial coefficient.

From the implicit expression for $\nu_{2,t}$ in (13) or the explicit expression in (14), we see also that the consumption shares of the two agents will depend on the ratio of the state-price densities in the single-agent economies:

$$\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} = \frac{\lambda_{1,0}}{\lambda_{2,0}} e^{(\beta_2-\beta_1)t} \frac{\xi_{1,t}}{\xi_{2,t}} \frac{H_{2,t}^{1-\gamma_2}}{H_{1,t}^{1-\gamma_1}} Y_t^{\gamma_2-\gamma_1}, \tag{16}$$

which from (16) depends on the differences in initial endowments $\lambda_{k,0}$, subjective discount rates β_k, beliefs $\xi_{k,t}$, sensitivities to the historical standard of living $H_{k,t}$, and risk aversions γ_k.

[12] See Abadir (1999) for an introduction to hypergeometric functions. Because the derivation of the sharing rule for the general case in which η is any real number is given in the Appendix, the derivation of the sharing rule in terms of hypergeometric functions when η is a natural number greater than or equal to five is not included.

2.2 Equilibrium state price density

We now give the level of the equilibrium state-price density using convergent series. Equation (17) in the proposition below shows that the equilibrium state-price density can be expressed as a *linear* combination of state-price densities of *single-agent* economies, that is, $\hat{\pi}_{k,t}, k \in \{1,2\}$, defined in (12), where the individual terms $\hat{\pi}_{k,t}$ solely depend on exogenous variables.

Proposition 2 (Equilibrium state-price density). The equilibrium state-price density is given by

$$\pi_t = \begin{cases} \sum_{n=0}^{\infty} a^{\pi}_{n,1} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R, \\ \sum_{n=0}^{\infty} a^{\pi}_{n,2} \hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} , \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \end{cases} \tag{17}$$

where R is defined in (15), $a^{\pi}_{n,1} = a^{\pi}_{n,2} = 1$ for $n=0$, and

$$a^{\pi}_{n,1} = \gamma_1 \frac{(-1)^{n+1}}{n} \binom{n\frac{\gamma_1}{\gamma_2} - \gamma_1 - 1}{n-1}, n \in \mathbb{N}, \tag{18}$$

$$a^{\pi}_{n,2} = \gamma_2 \frac{(-1)^{n+1}}{n} \binom{n\frac{\gamma_2}{\gamma_1} - \gamma_2 - 1}{n-1}, n \in \mathbb{N}. \tag{19}$$

To understand the above proposition, note that $\hat{\pi}_{1,t}$ and $\hat{\pi}_{2,t}$ are the state-price densities for the two *single-agent* economies. Then, $\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}$ is the state-price density of an *underlying economy*, constructed as the Hölder mean of the state price densities from two single-agent economies.[13] Finally, the equilibrium state-price density in (17) is a linear combination of the state-price densities of the underlying economies.

The expression for the equilibrium state-density in (17) can be simplified if agents have the same risk aversion, $\gamma_1 = \gamma_2 = \gamma$, and a further simplification is possible if γ is a natural number. These simpler expressions are given in the corollary that follows.

Corollary 1 (Equilibrium state price density with identical risk aversion). Suppose agents have identical risk aversion, that is, $\gamma_1 = \gamma_2 = \gamma$, but different beliefs. Then the equilibrium state-price density is given by

$$\pi_t = \begin{cases} \sum_{n=0}^{\infty} a^{\pi}_{n} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,t}^{\frac{n}{\gamma}}, & \hat{\pi}_{2,t} < \hat{\pi}_{1,t}, \\ \sum_{n=0}^{\infty} a^{\pi}_{n} \hat{\pi}_{1,t}^{\frac{n}{\gamma}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma}}, & \hat{\pi}_{2,t} > \hat{\pi}_{1,t}, \end{cases} \tag{20}$$

[13] If p is a nonzero real number, the Hölder mean of a and b with weights w and $1-w$, and exponent p is: $\left(wa^p + (1-w)b^p\right)^{\frac{1}{p}}$, and when $p \to 0$, the Hölder mean reduces to $a^w b^{1-w}$.

where, denoting the set of natural numbers that includes 0 by $\mathbb{N}_0$,

$$a_n^\pi = \binom{\gamma}{n}, n \in \mathbb{N}_0. \tag{21}$$

If relative-risk aversion, γ, is a natural number, then the equilibrium state-price density can be further simplified to a finite sum:

$$\pi_t = \sum_{n=0}^{\gamma} a_n^\pi \hat{\pi}_{1,t}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,t}^{\frac{n}{\gamma}} \tag{22}$$

$$= \left(\hat{\pi}_{1,t}^{\frac{1}{\gamma}} + \hat{\pi}_{2,t}^{\frac{1}{\gamma}} \right)^{\gamma}. \tag{23}$$

Thus, the expression for the equilibrium state-price density in (23) is a power mean (with exponent $\frac{1}{\gamma}$) of the individual agent state-price densities. It follows from well-known properties of the power mean that the state-price density in Equation (23) is increasing in relative-risk aversion, γ. The intuition for this is that the agents with higher risk aversion will be more willing to pay for a unit of consumption in a given state. If $\gamma = 1$, the power mean reduces to the arithmetic mean; if $\gamma \to \infty$, it reduces to the geometric mean; and, if $\gamma \to 0$, it reduces to the maximum of the individual-agent state-price densities.

The special case considered in Corollary 1, in which $\gamma_1 = \gamma_2 = \gamma$, with γ being a natural number, is similar to the model studied in Dumas, Kurshev, and Uppal (2009, their Equation (35)), where they obtain a similar expression for the state-price density. Because γ needs to be a natural number, this special case does not allow one to study the case of risk aversion smaller than one. Our Proposition 2, in contrast, allows for different risk aversion parameters for the two agents and does not restrict their values to be natural numbers.

2.3 Price-dividend ratio for dividend strips and equity

In this section, we first identify the price of a dividend strip, which is a claim that pays a single cash flow of Y_u at a particular time u.[14] Empirical properties of dividend strips of different maturities have been studied in van Binsbergen, Brandt, and Koijen (2012), van Binsbergen et al. (forthcoming), and Boguth et al. (2012). This empirical analysis of dividend strips is related to the work in Hansen, Heaton, and Li (2008) and Lettau and Wachter (2011), who study the term structure of returns on the aggregate market, along with the risk and return characteristics of value and growth stocks. See also Hansen and Scheinkman (2008) and Hansen (2012), who develop an approach based on valuation operators to study the term structure of the risk-return trade-off.

[14] In the Appendix, we show how to value more general cash flow payments.

We denote the date-t price of this dividend strip by $V_{t,u}^{Y}$, where

$$V_{t,u}^{Y} = Y_t v_{t,u}^{Y},$$

and where the price-dividend ratio of the dividend strip is

$$v_{t,u}^{Y} = E_t\left[\frac{\pi_u}{\pi_t}\frac{Y_u}{Y_t}\right]. \tag{24}$$

We can then obtain the price of equity by integrating the price of dividend strips over time. To see this, let P_t^Y denote the price of the single share of the risky asset (stock), which is a claim on the aggregate dividend, Y_t. The stock price is given by

$$P_t^Y = Y_t p_t^Y,$$

where the price-dividend ratio for equity p_t^Y is:

$$p_t^Y = E_t \int_t^{\infty} \frac{\pi_u}{\pi_t}\frac{Y_u}{Y_t} du = \int_t^{\infty} v_{t,u}^{Y} du. \tag{25}$$

We derive a representation for the price-dividend ratio of the dividend strip, $v_{t,u}^{Y}$, by using the state-price density in Proposition 2. Because the state-price density is one of two linear combinations of state-price densities from a set of underlying economies, depending on whether $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} \gtrless R$, the price-dividend ratio of the dividend strip, $v_{t,u}^{Y}$, is a sum of two weighted averages. The first is a weighted average of price-dividend ratios from a set of underlying economies conditional on $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R$, and the second is a weighted average of price-dividend ratios from a set of underlying economies conditional on $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R$.

Proposition 3 (Price of dividend strip). The time-t price of the dividend strip, which pays the cash flow Y_u at date $u > t$, is given by $V_{t,u}^{Y} = v_{t,u}^{Y} Y_t$, where

$$v_{t,u}^{Y} = \sum_{n=0}^{\infty} \omega_{n,1,t} \phi_{n,1,t,u}^{Y} + \sum_{n=0}^{\infty} \omega_{n,2,t} \phi_{n,2,t,u}^{Y}, \tag{26}$$

where the weights $\omega_{n,1,t}, n \in \mathbb{N}_0$, and $\omega_{n,2,t}, n \in \mathbb{N}_0$, are given by

$$\omega_{n,1,t} = a_{n,1}^{\pi} (\nu_{1,t}^{\gamma_1})^{1-\frac{n}{\gamma_2}} (\nu_{2,t}^{\gamma_2})^{\frac{n}{\gamma_2}}, n \in \mathbb{N}_0 \tag{27}$$

$$\omega_{n,2,t} = a_{n,2}^{\pi} (\nu_{1,t}^{\gamma_1})^{\frac{n}{\gamma_1}} (\nu_{2,t}^{\gamma_2})^{1-\frac{n}{\gamma_1}}, n \in \mathbb{N}_0, \tag{28}$$

and each set of weights sums to one:

$$\sum_{n=0}^{\infty} \omega_{n,1,t} = \sum_{n=0}^{\infty} \omega_{n,2,t} = 1, \tag{29}$$

and $\phi^Y_{n,1,t,u}$ ($\phi^Y_{n,2,t,u}$) is the price-dividend ratio for the "spanning asset" which pays the cashflow Y_u at date u, provided $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R$ $\left(\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right)$,

$$\phi^Y_{n,1,t,u} = E_t\left[\frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}\right], n \in \mathbb{N}_0, \tag{30}$$

$$\phi^Y_{n,2,t,u} = E_t\left[\frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}\right], n \in \mathbb{N}_0. \tag{31}$$

The above proposition is useful because we see from (26) that we have reduced the problem of finding the value of dividend strips for the heterogeneous-agent economy to the problem of finding the value of the assets $\phi^Y_{n,1,t,u}$ and $\phi^Y_{n,2,t,u}$ in the underlying economies, with state-price densities $\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}$ and $\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}$, respectively. In other words, from (26) we can see that the value of a dividend strip is spanned by the asset values, $\phi^Y_{n,1,t,u}$ and $\phi^Y_{n,2,t,u}$, leading us to call them "*spanning assets*." Furthermore, by virtue of (25), we can see that the same spanning assets can be used to value equity; that is the price-dividend ratio for equity is:

$$p^Y_t = \sum_{n=0}^{\infty}\omega_{n,1,t}\int_t^{\infty}\phi^Y_{n,1,t,u}du + \sum_{n=0}^{\infty}\omega_{n,2,t}\int_t^{\infty}\phi^Y_{n,2,t,u}du.$$

The power of this result rests on the fact that computing price-dividend ratios directly using (24) or (25) involves solving a valuation problem with an endogenous state variable $\nu_{1,t}$, whereas computing price-dividend ratios indirectly via Proposition 3 eliminates the endogenous state variable because the values of the spanning assets are determined in the underlying economies in which the state-price density is the Hölder mean of the state-price density of homogeneous-agent economies.

We now consider two special cases: the first where the two agents have the same risk aversion, $\gamma_1 = \gamma_2 = \gamma$, and the second, where the two agents have the same risk aversion and γ is a natural number.

Corollary 2 (Price of dividend strip with identical risk aversion). When risk aversions are identical, $\gamma_1 = \gamma_2 = \gamma$, then

$$v^Y_{t,u} = \sum_{n=0}^{\infty}\omega_{n,1,t}\phi^Y_{n,1,t,u} + \sum_{n=0}^{\infty}\omega_{n,2,t}\phi^Y_{n,2,t,u},$$

where

$$\omega_{n,1,t} = \binom{\gamma}{n}(\nu^\gamma_{1,t})^{1-\frac{n}{\gamma}}(\nu^\gamma_{2,t})^{\frac{n}{\gamma}}, n \in \mathbb{N}_0 \tag{32}$$

$$\omega_{n,2,t} = \binom{\gamma}{n}(\nu^\gamma_{1,t})^{\frac{n}{\gamma}}(\nu^\gamma_{2,t})^{1-\frac{n}{\gamma}}, n \in \mathbb{N}_0, \tag{33}$$

and each set of weights sums to one: $\sum_{n=0}^{\infty}\omega_{n,1,t}=\sum_{n=0}^{\infty}\omega_{n,2,t}=1$.

If in addition to risk aversions being identical, $\gamma_1=\gamma_2=\gamma$, we also have that $\gamma\in\mathbb{N}$, then the above expressions simplify further to:

$$v_t^Y=\sum_{n=0}^{\gamma}\omega_{n,t}\,v_{n,t,u}^Y, \tag{34}$$

where

$$v_{n,t,u}^Y=E_t\left[\frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma}}\,\hat{\pi}_{2,u}^{\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}\,\hat{\pi}_{2,t}^{\frac{n}{\gamma}}}\frac{Y_u}{Y_t}\right], n\in\mathbb{N}_0 \text{ and } n\leq\gamma,$$

$$\omega_{n,t}=\begin{pmatrix}\gamma\\ n\end{pmatrix}\left(\nu_{1,t}^{1-\frac{n}{\gamma}}\,\nu_{2,t}^{\frac{n}{\gamma}}\right)^{\gamma}, \tag{35}$$

and $\sum_0^{\gamma}\omega_{n,t}=1$.

From (34), we see that the price-dividend ratio of the dividend strip in the economy with heterogenous beliefs is a weighted sum of the price-dividend ratios of the dividend strips in $1+\gamma$ underlying economies, where in the n'th such economy, the state-price density, $\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}\,\hat{\pi}_{2,t}^{\frac{n}{\gamma}}$, is the Hölder mean of the state-price densities in the single-agent economies. The special case considered in Corollary 2 is similar to the model studied by Yan (2008, his Proposition 3), where he obtains closed-form results for only the case in which the risk aversion parameter γ is identical across agents and γ is a natural number, which then excludes the case of risk aversion smaller than one. Our Proposition 3, in contrast, allows for different risk aversion parameters for the two agents and does not restrict their values to be natural numbers.

Below, in Proposition 8 of Section 4, we derive explicit expressions for $\phi_{n,1,t,u}^Y$ and $\phi_{n,2,t,u}^Y$ by making particular assumptions for the dynamics of the aggregate endowment process and beliefs. In Section 5, we show how to characterize $\phi_{n,1,t,u}^Y$ and $\phi_{n,2,t,u}^Y$ for affine processes.

3. Dynamics of the Consumption Share and State-Price Density

So far in our analysis, we have not specified particular processes for aggregate dividends and beliefs. However, to characterize the dynamics of the consumption share, the state-price density, and asset prices, we need to specify the dynamics for aggregate dividends and beliefs of the two agents, which we do in Section 3.1. Next, we define aggregate preference parameters and aggregate beliefs in Section 3.2. Then, in Section 3.3 we derive the implications of heterogeneity for the dynamics of the sharing rule and the dynamics of the state-price density.

3.1 The processes for aggregate endowment and beliefs of the two agents

The true evolution of the aggregate endowment, Y, is assumed to be:

$$\frac{dY_t}{Y_t} = \mu_Y\, dt + \sigma_Y\, dZ_t, \quad Y_0 > 0, \tag{36}$$

in which μ_Y and σ_Y are constants and Z is a one-dimensional Brownian motion.

Agent k believes that the expected growth rate of the endowment process takes the constant value, $\mu_{Y,k}$, as in Yan (2008), Borovička (2012), and Fedyk, Heyerdahl-Larsen, and Walden (2012). Therefore, defining the quantity $\sigma_{\xi,k} \equiv \frac{\mu_{Y,k}-\mu_Y}{\sigma_Y}$, Agent k's beliefs can be represented by an exponential martingale

$$\xi_{k,t} = e^{-\frac{1}{2}\sigma_{\xi,k}^2 t + \sigma_{\xi,k} Z_t}, \tag{37}$$

which can be written as: [15]

$$\frac{d\xi_{k,t}}{\xi_{k,t}} = \sigma_{\xi,k}\, dZ_t. \tag{38}$$

Hence, by Girsanov's theorem, Agent k believes that the process for aggregate endowments is

$$\frac{dY_t}{Y_t} = \mu_{Y,k}\, dt + \sigma_Y\, dZ_{k,t},$$

where $Z_{k,t} = Z_t - \sigma_{\xi,k} t$ is a standard Brownian motion under $\mathbb{P}^k$, which represents Agent k's beliefs. Hence, we see that the expected growth rate of the aggregate endowment under $\mathbb{P}^k$ is $\mu_{Y,k}$.[16]

We quantify the *level of disagreement* between the two agents via the process, ξ_t, where $\xi_t \equiv \frac{\xi_{2,t}}{\xi_{1,t}} = e^{-\frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2)t + (\sigma_{\xi,2}-\sigma_{\xi,1})Z_t}$, and its dynamics are

$$\frac{d\xi_t}{\xi_t} = \mu_\xi\, dt + \sigma_\xi\, dZ_t,$$

$$\text{where} \qquad \mu_\xi \equiv -\sigma_{\xi,1}(\sigma_{\xi,2} - \sigma_{\xi,1}),$$

$$\sigma_\xi \equiv (\sigma_{\xi,2} - \sigma_{\xi,1}) = \frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y}.$$

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15 The exponential martingale, $\xi_{k,t}$, defines the probability measure $\mathbb{P}^k$ on $(\Omega, \mathcal{F})$, via $\mathbb{P}^k(e_T) = E_t[1_{e_T}\xi_{k,T}]$, $\forall t, T \in [0,\infty)$, $t \le T$, where e_T is an event that occurs at time T, and $\mathbb{P}^k(e_T)$ is the probability of its occurrence based on information known at time t.

16 Note that the probability measures $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}$ are all equivalent; that is, they agree on which events are impossible.

3.2 Definitions of aggregate preference parameters and beliefs

In this section, we define the aggregate preferences in the economy and also aggregate beliefs, which will then be used in the expressions for the dynamics of the sharing rule and the dynamics of the state price density.

Definition 1 (Aggregate risk aversion). The aggregate relative-risk aversion, $\mathbf{R}_t$, in the economy is defined as the consumption-share-weighted harmonic mean of individual agents' relative-risk aversions:

$$\mathbf{R}_t = \left(\nu_{1,t} \frac{1}{\gamma_1} + \nu_{2,t} \frac{1}{\gamma_2} \right)^{-1}. \tag{39}$$

Equivalently, the *aggregate risk tolerance* in the economy, $1/\mathbf{R}_t$, is the consumption-share-weighted arithmetic mean of individual agents' risk tolerances, $1/\gamma_k$.

Defining w_k to be the consumption-share weighted relative-risk tolerances of investor k:

$$w_k = \frac{\frac{1}{\gamma_k} \nu_{k,t}}{\frac{1}{\gamma_1} \nu_{1,t} + \frac{1}{\gamma_2} \nu_{2,t}}, \text{ and } w_1 + w_2 = 1, \tag{40}$$

we can then define the aggregate rate of time preference, aggregate sensitivity of the risk-free rate to the growth rate of aggregate consumption, aggregate prudence, and aggregate beliefs as follows.

Definition 2 (Aggregate rate of time preference). The *aggregate rate of time preference* in the economy, $\boldsymbol{\beta}_t$, is given by the weighted arithmetic mean of individual agents' rates of time preference, where the weights are the consumption-share weighted relative-risk tolerances of the two investors given in Equation (40):

$$\boldsymbol{\beta}_t = w_{1,t} \beta_1 + w_{2,t} \beta_2.$$

Definition 3 (Aggregate sensitivity of risk-free rate). The aggregate sensitivity of the risk-free rate to the growth rate of aggregate consumption in the steady state is given by the following harmonic mean of the sensitivities of the risk-free rate to the growth rate of aggregate consumption in homogeneous agent economies:

$$\boldsymbol{\psi}_t = \left(w_{1,t} \frac{1}{\psi_1} + w_{2,t} \frac{1}{\psi_2} \right)^{-1},$$

where the weights are the consumption-share weighted relative-risk tolerances of the two investors, as defined in (40).

Definition 4 (Aggregate prudence). The quantity $\mathbf{P}_t$ is the *aggregate prudence* in the economy:[17]

$$\mathbf{P}_t = (1+\gamma_1)\left(\frac{\mathbf{R}_t}{\gamma_1}\right)^2 \nu_{1,t} + (1+\gamma_2)\left(\frac{\mathbf{R}_t}{\gamma_2}\right)^2 \nu_{2,t}.$$

Definition 5 (Aggregate belief). The *aggregate belief*, $\boldsymbol{\mu}_{Y,t}$, is given by the weighted arithmetic mean of the beliefs of individual agents,

$$\boldsymbol{\mu}_{Y,t} = w_{1,t}\,\mu_{Y,1} + w_{2,t}\,\mu_{Y,2},$$

where the weights are the consumption-share weighted relative-risk tolerances of the two investors, as defined in Equation (40).

3.3 Dynamics of the consumption-sharing rule

We are now ready to describe the evolution of the consumption-sharing rule.

Proposition 4 (Dynamics of the sharing rule). The true evolution of the sharing rule is given by:

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}}dt + \sigma_{\nu_{1,t}}dZ_t,$$

where

$$\sigma_{\nu_{1,t}} = \frac{\nu_{2,t}\mathbf{R}_t}{\gamma_1\gamma_2}\left[\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y} + (\gamma_2-\gamma_1)\sigma_Y\right], \tag{41}$$

$$\mu_{\nu_{1,t}} = \nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Bigg\{\beta_2-\beta_1+\left(\frac{\frac{\mu_{Y,1}+\mu_{Y,2}}{2}-\mu_Y}{\sigma_Y}\right)\left(\frac{\mu_{Y,2}-\mu_{Y,1}}{\sigma_Y}\right) \tag{42}$$

$$+\left(\frac{1}{\psi_2}-\frac{1}{\psi_1}\right)\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right)+\left[\left(\gamma_2-\frac{1}{\psi_2}\right)-\left(\gamma_1-\frac{1}{\psi_1}\right)\right]\lambda_x(\bar{\omega}-\omega_t) \tag{43}$$

$$+\frac{1}{2}\frac{\gamma_2 w_{2,t}^2-\gamma_1 w_{1,t}^2}{\gamma_1\gamma_2}\left[\left(\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}\right)^2+2(\mu_{Y,1}-\mu_{Y,2})(\gamma_2-\gamma_1)+(\gamma_2-\gamma_1)^2\sigma_Y^2\right]\Bigg\}. \tag{44}$$

[17] Note that aggregate prudence may be larger than the prudence of either agent; that is, aggregate prudence is not necessarily bounded between the prudence of the individual agents. Consequently, the interest rate in the two-agent economy, which depends on aggregate prudence as shown in Equation (49), may not be bounded between the interest rates in the economies with only one of the two agents, as observed in Wang (1996).

From (41), we see that the volatility of the sharing rule, $\sigma_{\nu_{1,t}}$, is driven by differences in risk aversion and differences in beliefs, but not differences in subjective discount rates or habit, which have only a deterministic effect and so appear only in the expression for the drift of the sharing rule, $\mu_{\nu_{1,t}}$. The expression for $\sigma_{\nu_{1,t}}$ in (41) shows that, if agents have identical beliefs, then an increase in heterogeneity in risk aversion leads to an increase in the volatility of the consumption share of Agent 1 because of an increase in consumption risk sharing. Similarly, if agents have identical risk aversions ($\gamma_1 = \gamma_2$), then an increase in heterogeneity in beliefs leads to an increase in the volatility of the consumption share of Agent 1.[18]

However, when both risk aversion and beliefs are heterogeneous, then the effect of an increase in the heterogeneity in either one of these factors on the volatility of the consumption share depends on whether it reinforces or counteracts the effect of the other factor. From (41) we observe that $\sigma_{\nu_{1,t}} > 0$ if and only if

$$\gamma_2 - \gamma_1 > \frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y^2}; \tag{45}$$

that is, if the more risk-averse agent is not too optimistic relative to the less risk-averse agent. If this condition is satisfied, then we see from the definition of aggregate risk aversion in (39) that $\mathbf{R}_t$ will be countercyclical, because when the aggregate endowment has a positive shock, the weight on the risk aversion of Agent 1 increases, and so the aggregate risk aversion in the economy decreases. Therefore, the heterogeneity in risk aversion and beliefs can generate countercyclical aggregate risk aversion endogenously. Moreover, if Agent 2, who has the higher risk aversion, is also the more pessimistic agent, then the heterogeneity in beliefs reinforces the effect arising from heterogeneity in risk aversion. This countercyclical behavior of aggregate risk aversion has been previously recognized in the multiagent models of Chan and Kogan (2002) and Xiouros and Zapatero (2010), where agents have heterogeneous risk aversions but homogeneous beliefs, and this feature appears in the single-agent model of Campbell and Cochrane (1999) as a consequence of the assumption of habit-formation.

Equation (42) shows how $\mu_{\nu_{1,t}}$ depends on differences in subjective discount rates and differences in beliefs. The impact of differences in ψ_k is given in (43). We also see how $\mu_{\nu_{1,t}}$ is affected by the volatility of aggregate endowment growth, σ_Y, and the differences in beliefs, both of which appear in (44).[19]

[18] In the case in which agents have different risk aversion but the same beliefs, $\sigma_{\nu_{1,t}}$ is always positive.

[19] The discussion above illustrates the benefit of having the closed-form results in Propositions 1 and 4. Because we have explicit expressions for the sharing rule and its dynamics, we can understand exactly how these are affected by the parameters for preferences, beliefs, and the endowment process.

3.4 Dynamics of state-price density: Risk-free rate and market price of risk

In this section, we determine the dynamics of the state-price density, and hence, the equilibrium risk-free rate and market price of risk.

The central planner's state-price density, π_t, is given by

$$\pi_t = \hat{\pi}_{k,t}\nu_{k,t}^{-\gamma_k} = \xi_{k,t}\pi_{k,t}, \quad k \in \{1,2\}, \tag{46}$$

where $\hat{\pi}_k$ is the state-price density in a homogeneous-agent economy in which all agents are of type k and is defined in (12), whereas π_k is the state-price density of Agent k in the heterogeneous-agent economy, and is defined in (8).[20] From standard results in asset pricing (see Duffie 2001, Section 6.D), the evolution of the central planner's state-price density π_t is:

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \theta_t dZ_t, \tag{47}$$

and the evolution of Agent k's state-price density, $\pi_{k,t}$, is:

$$\frac{d\pi_{k,t}}{\pi_{k,t}} = -r_t dt - \theta_{k,t} dZ_{k,t}, \tag{48}$$

where, if B_t denotes the price of a locally risk-free bond in zero net supply, then, the risk-free return r_t is given by $\frac{dB_t}{B_t} = r_t dt$.

Note from (48) that each agent has her own market price of risk, θ_k; however, because the instantaneously risk-free bond is a traded security, the two agents must agree on its price, and hence, on the interest rate.

3.4.1 The risk-free interest rate. The following proposition gives the expression for the risk-free rate.

Proposition 5 (Locally risk-free interest rate). The locally risk-free rate is given by:

$$r_t = \boldsymbol{\beta}_t + \mathbf{R}_t \boldsymbol{\mu}_{Y,t} - \frac{1}{2}\mathbf{R}_t \mathbf{P}_t \sigma_Y^2 - \left(\mathbf{R}_t - \frac{1}{\boldsymbol{\psi}_t}\right)\lambda_x \omega_t \tag{49}$$

$$+\frac{1}{2} w_{1,t} w_{2,t}\left(1 - \frac{\mathbf{R}_t}{\gamma_1\gamma_2}\right)\sigma_\xi^2 - w_{1,t} w_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(\mu_{Y,1} - \mu_{Y,2}), \tag{50}$$

in which ω_t is defined in (3), $\bar{\omega}_k = \frac{\mu_{Y,k} - \frac{1}{2}\sigma_Y^2}{\lambda_x}$ is the long-run mean of $\omega_t \equiv y_t - x_t$ under the beliefs of Agent k, that is, under the probability measure $\mathbb{P}^k$, and the weights w_k are defined in (40).

[20] Because financial markets are effectively complete, marginal utilities of consumption are equal across agents for each state, and therefore the first-order condition for consumption in (10) ensures that the expression in (46) is the same for $k \in \{1,2\}$.

The corollary below gives the risk-free rate for some special cases.

Corollary 3 (Locally risk-free interest rate: Special cases). If agents have identical and correct beliefs, then the locally risk-free rate is given by

$$r_t = \boldsymbol{\beta}_t + \mathbf{R}_t \mu_Y - \frac{1}{2}\mathbf{R}_t \mathbf{P}_t \sigma_Y^2 - \left(\mathbf{R}_t - \frac{1}{\boldsymbol{\psi}_t}\right)\lambda_x \omega_t. \tag{51}$$

On the other hand, if $\gamma_1 = \gamma_2 = \gamma$ and $\psi_1 = \psi_2 = \psi$, but agents have different beliefs and rates of time preference, then the locally risk-free rate is given by

$$r_t = \sum_{k=1}^{2} \nu_{k,t}\beta_k + \gamma \sum_{k=1}^{2} \nu_{k,t}\mu_{Y,k} - \frac{1}{2}\,\gamma(1+\gamma)\sigma_Y^2 - \left(\gamma - \frac{1}{\psi}\right)\lambda_x \omega_t \tag{52}$$
$$+ \frac{1}{2}\nu_{1,t}\nu_{2,t}\left(1 - \frac{1}{\gamma}\right)\sigma_\xi^2.$$

To interpret the expression for the interest rate, recall that in a standard Lucas (1978) economy in which all agents have correct and identical beliefs and identical preferences that are given by a power function, the expression for the interest rate is

$$\hat{r}_k = \beta_k + \gamma_k \mu_Y - \frac{1}{2}\gamma_k(1+\gamma_k)\sigma_Y^2. \tag{53}$$

From Equation (53), we see that the interest rate is positively related to the rate of impatience, β_k, positively related to the growth rate of aggregate endowment, μ_Y, and scaled by risk aversion γ_k; and the third term arises because of precautionary savings in the face of aggregate endowment risk, which leads to a drop in the interest rate, where the magnitude of the drop depends on $(1+\gamma_k)$, the relative prudence of agents, and on risk aversion, γ_k.

When investors are identical but their preferences exhibit habit and their beliefs $\mu_{Y,k}$ are allowed to deviate from the true growth rate μ_Y, then the interest rate is given by

$$\hat{r}_{k,t} = \beta_k + \gamma \mu_{Y,k} - \frac{1}{2}\gamma_k(1+\gamma_k)\sigma_Y^2 - \left(\gamma_k - \frac{1}{\psi_k}\right)\lambda_x \omega_t. \tag{54}$$

Comparing (54) with (53) we see that the first term, β_k, is the same in both expressions; in the second term, the belief of each agent about the growth rate of aggregate endowments, $\mu_{Y,k}$, replaces the true growth rate, μ_Y; the third term, which reflects the effect of the precautionary savings, is the same; and, the fourth term is new, and it reflects the effect of habit.

Equation (51) of the corollary shows that if only risk aversions are heterogeneous but beliefs are homogeneous and correct, then the risk-free rate has the same form as in (54) for a single-agent economy, but with the aggregate quantities $\boldsymbol{\beta}_t$, $\mathbf{R}_t$, and aggregate prudence, $\mathbf{P}_t$, replacing their single-agent counterparts; note, however, that because the weights used to construct these aggregate measures vary over time, the aggregate measures will be time varying rather than constant.

On the other hand, if only beliefs are heterogeneous but preferences are homogeneous, then we see from the last term in (52) that if $\gamma < 1$ the differences in beliefs will decrease the interest rate, or equivalently, increase the price of the instantaneously riskless bond. This effect is similar to the premium ("bubble") in asset prices that has been studied in Harrison and Kreps (1978) and Scheinkman and Xiong (2003) for the case of risk neutrality ($\gamma = 0$) in the presence of short sale constraints; over here, we get a similar effect for agents who are risk averse without needing to constrain short sales. However, if $\gamma > 1$, then the price of the bond *decreases* with heterogeneity in beliefs, an observation also made in Dumas, Kurshev, and Uppal (2009).

The risk-free rate is given in lines (49) and (50) for the case in which agents have both heterogeneous beliefs and preferences. The first three terms on line (49) correspond to the three terms in (51), and are related to the subjective time preference of the two agents, the growth rate of aggregate endowment, and the demand for precautionary savings. The last term on line (49) arises because of habit. The two terms on line (50) arise because of differences in beliefs. The first term in (50) increases the risk-free rate when the aggregate risk aversion is less than the square of the geometric mean of risk aversion; that is, $\mathbf{R}_t < \gamma_1\gamma_2$, which is true if and only if $\gamma_1 > 1$.[21] It follows that if $\gamma_1 > 1$ ($\gamma_1 < 1$), then heterogeneity in beliefs increases (decreases) the risk-free rate. The second term arises because of differences in both risk aversion and in beliefs; that is, $\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(\mu_{Y,1} - \mu_{Y,2})$. When the less risk-averse agent is also the more optimistic agent, that is, $\mu_{Y,1} > \mu_{Y,2}$, this term decreases the risk-free rate.

One of the limitations of the representative-agent general-equilibrium model of asset pricing with power utility is that, when risk aversion is increased in order to improve the match of the equity risk premium in the model to that in the data, the risk-free interest rate in the model becomes too high relative to the data; this is the "risk-free rate puzzle" identified in Weil (1989). From the discussion above, we see that both heterogeneity in beliefs and preferences have the potential to influence the interest rate.

3.4.2 The market price of risk. From (47), we see that the volatility of the central planner's state-price density is given by the market price of risk, θ_t, whereas from (48) we see that the volatility of the state price density for each individual agent is given by the perceived market price of risk, $\theta_{k,t}$. The following proposition gives the expressions for these market prices of risk.

21 Note that because $\mathbf{R}_t \leq \gamma_2$, $\mathbf{R}_t < \gamma_1\gamma_2$ if and only if $\gamma_1 > 1$.

Proposition 6 (Market price of risk). The market price of risk of the central planner, θ_t, is:

$$\theta_t = \mathbf{R}_t\,\sigma_Y + \frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}, \tag{55}$$

and the market price of risk perceived by Agent k is:

$$\theta_{k,t} = \mathbf{R}_t\,\sigma_Y + \frac{\mu_{Y,k} - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}, k \in \{1,2\}. \tag{56}$$

The corollary that follows gives the market prices of risk for the central planner and the two agents for the special cases in which agents have identical preferences or identical beliefs.

Corollary 4 (Market price of risk: Special cases). If agents have identical and correct beliefs, then the central planner's market price of risk, θ_t, and the market price of risk perceived by Agent k, $\theta_{k,t}$ are given by:

$$\theta_t = \theta_{k,t} = \mathbf{R}_t \sigma_Y, k \in \{1,2\}. \tag{57}$$

On the other hand, if agents have identical relative-risk aversion, $\gamma_1 = \gamma_2 = \gamma$, but different beliefs, rates of time preference, and ψ_k, then the central planner's equilibrium market price of risk is

$$\theta_t = \gamma\,\sigma_Y + \frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y},$$

and the market price of risk perceived by Agents k is

$$\theta_{k,t} = \gamma\sigma_Y + \frac{\mu_{Y,k} - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}. \tag{58}$$

To understand the expressions for the market price of risk in the above corollary and proposition, note that in an economy in which all agents have correct and identical beliefs, and identical risk aversion, $\gamma_1 = \gamma_2 = \gamma$, the market price of risk is $\theta = \gamma\,\sigma_Y$. If only preferences are different across agents, then γ is replaced by the average risk aversion in the economy, $\mathbf{R}_t$, and the market price of risk is given by (57), with both agents agreeing with this market price of risk. On the other hand, if preferences are identical but beliefs are heterogeneous, then we see that agents do not agree on the market price of risk. From (58) we see that if Agent 2 is pessimistic relative to Agent 1, $\mu_{Y,1} > \mu_{Y,2}$, then the market price of risk perceived by Agent 1 will be increased. The magnitude of this increase depends on the consumption-share of Agent 2, $\nu_{2,t}$, because this determines Agent 2's influence on equilibrium stock market returns.

For the general case in (56), in which both beliefs and risk aversions are different, we see that the increase in the market price of risk perceived by Agent 1 will depend on the consumption share of Agent 2, $\nu_{2,t}$, and the agent's risk tolerance, $1/\gamma_2$, relative to aggregate risk tolerance in the economy, $1/\mathbf{R}_t$, because these are the two factors that determine the size of the position Agent 2 takes in the stock market. Finally, from the expression in (55) for the general case in which there is heterogeneity in both preferences and beliefs, we see that the market price of risk for the central planner will increase if average beliefs are pessimistic; that is, $\mu_Y > \boldsymbol{\mu}_{Y,t}$. The intuition for this is that, if agents are pessimistic on average, then the compensation for bearing risk must be relatively higher than what it needs to be in an economy in which agents have the correct average beliefs.

Note also that the market price of risk is countercyclical in the data and in the model of Campbell and Cochrane (1999). This will be true also in our model if $\mathbf{R}_t$ is countercyclical, which requires that the more risk-averse agent not be too optimistic relative to the less risk-averse agent—this exact condition is given in Equation (45).

3.5 Survival of the two agents

In this section, we derive the conditions under which both agents survive in the long run. We say that the economy is stationary if both agents survive. To formalize the concept of survival, we introduce the concept of *almost-sure (a.s.) survival* with respect to a particular probability measure, as in Kogan et al. (2006) and Yan (2008).

Definition 6. Agent k survives $\mathbb{P}$-a.s. if

$$\lim_{t\to\infty} \nu_{k,t} > 0, \mathbb{P}\text{-a.s.}$$

Note that if an agent's consumption share is strictly above zero with a probability of less than one, under $\mathbb{P}$ say, then she does not survive $\mathbb{P}$-almost surely. Furthermore, the probability measure is important, because an agent may believe she survives almost surely (with respect to the probability measure representing her beliefs), when in fact, she almost surely does not survive under the true probability measure $\mathbb{P}$.[22]

The following proposition provides a condition for almost sure survival under $\mathbb{P}$, extending the results in Yan (2008) along two dimensions: introducing preferences with external habit and allowing for heterogeneity in relative-risk aversion and habit.

Definition 7 (Survival index). Agent k has a survival index I_k, defined by

$$I_k = \beta_k + \frac{1}{2}\left(\frac{\mu_{Y,k} - \mu_Y}{\sigma_Y}\right)^2 + \frac{1}{\psi_k}\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right), k \in \{1,2\}. \quad (59)$$

It is important to note that relative-risk aversion, γ_k, does not affect the survival index. This is because of the effect of habit, as explained in Chan and Kogan (2002).

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[22] Agent k survives $\mathbb{P}^j$-a.s. if $\lim_{t\to\infty} \nu_{k,t} > 0, \mathbb{P}^j$-a.s.

Proposition 7 (Condition for almost-sure stationarity). The economy is almost surely stationary under $\mathbb{P}$ if and only if the following two conditions are satisfied:

$$\text{Condition 1:} \qquad I_1 = I_2;$$

$$\text{Condition 2:} \qquad \mu_{Y,1} - \frac{1}{\psi_1}\sigma_Y^2 = \mu_{Y,2} - \frac{1}{\psi_2}\sigma_Y^2.$$

Under Condition 1, the probability density function of Agent 1's consumption share at date $t+u$, conditional on date t information, has all its mass concentrated at 0 and 1, when $u \to \infty$ (see Proposition A1). Hence, each agent has a probability of surviving equal to 1/2. This situation arises, because the standard Brownian motion for the endowment process will eventually become extremely large or extremely small (because $\limsup_{t\to\infty} Z_t = \infty$ and $\liminf_{t\to\infty} Z_t = -\infty$). When this Brownian motion is extremely large, one of the agents will dominate the economy, and when the Brownian motion is extremely small, the other agent will dominate. Imposing Condition 2 in addition to Condition 1 rules out this behavior.

4. Asset Prices

In this section, we derive the stock price, the equity risk premium, the volatility of stock market returns, and the term structure of interest rates for the particular aggregate endowment process and beliefs specified in Equations (36) and (37), respectively, and then examine how these quantities are influenced by heterogeneity in beliefs, rates of time preference, habit, and risk aversion. In Section 5, we extend these pricing results to the case in which the logarithm of aggregate endowment and agents' beliefs follow general affine processes, as opposed to the particular processes assumed in Equations (36) and (37).

4.1 Valuation of equity

Recall from Equation (26) in Proposition 3 that the date-t price of a dividend strip which pays out Y_u units of consumption at date $u > t$, denoted by $V_{t,u}^Y$, can be obtained from the prices of the spanning assets. For the aggregate endowment and beliefs, given in Equations (36) and (37), the value of the spanning assets $\phi_{n,k,t,u}^Y$ can be expressed analytically. Denote the cumulative standard normal distribution function by $\Phi(z) = \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$. We then have the following result.

Proposition 8 (Value of spanning assets $\phi_{n,1,t,u}^Y$ and $\phi_{n,2,t,u}^Y$). Given the aggregate endowment process specified in Equation (36) and the beliefs process in (37), the values of $\phi_{n,1,t,u}^Y$ and $\phi_{n,2,t,u}^Y$ that pay Y_u at u whenever

$\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R$ and $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R$, respectively, are:

$$\phi^Y_{n,1,t,T} = v^{Y,hom}_{t,u}(k_1,b_1) e^{a_1\left[\hat{\mu}^Y_{q,1}(T-t)+\frac{1}{2}a_1\sigma_q^2(T-t)+b_1\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y\right]}\Phi\left(\frac{B^Y_1(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right), \tag{60}$$

$$\phi^Y_{n,2,t,T} = v^{Y,hom}_{t,u}(k_2,b_2) e^{a_2\left[\hat{\mu}^Y_{q,2}(T-t)+\frac{1}{2}a_2\sigma_q^2(T-t)+b_2\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y\right]}\Phi\left(\frac{\rho-B^Y_2(q_t,\omega_t,T-t)}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right), \tag{61}$$

where

$$
\begin{aligned}
v^{Y,hom}_{t,u}(k_j,b_j) &= e^{-k_j(T-t)+b_j(\omega_t-\hat{\omega}^Y_j)[1-e^{-\lambda_x(T-t)}]+\frac{1}{2}b_j^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2},\\
\hat{k}_j &= \beta_j + \frac{1}{\psi_j}\mu_{Y,j} - \frac{1}{2}\frac{1}{\psi_j}\left(1+\frac{1}{\psi_j}\right)\sigma_Y^2 + \frac{1}{\psi_j}\sigma_Y^2 - \mu_{Y,j},\\
a_1 &= -\frac{n}{\gamma_2},\; a_2 = \frac{n}{\gamma_1},\\
b_1 &= -\left[\frac{n}{\gamma_2}\left(\gamma_2 - \frac{1}{\psi_2}\right) + \left(1-\frac{n}{\gamma_2}\right)\left(\gamma_1 - \frac{1}{\psi_1}\right)\right],\\
b_2 &= -\left[\left(1-\frac{n}{\gamma_2}\right)\left(\gamma_2 - \frac{1}{\psi_2}\right) + \frac{n}{\gamma_1}\left(\gamma_1 - \frac{1}{\psi_1}\right)\right],\\
\rho &= \ln R.
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
A_k(q_t,\omega_t,T-t) &= \frac{1}{2}\left[\sigma_q^2(T-t) + 2d_\omega\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y + d_\omega^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2\right],\\
B^Y_k(q_t,\omega_t,T-t) &= q_t + \hat{\mu}^Y_{k,q}(T-t) + d_\omega[e^{-\lambda_x(T-t)}\omega_t + (1-e^{-\lambda_x(T-t)})\hat{\omega}^Y_k],\\
&\quad + a_k\sigma_q^2(T-t) + (b_k + a_k d_\omega)\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y + b_k d_\omega\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2,\\
\hat{\mu}^Y_{k,q} &= \mu_q + \left(\left(1-\frac{1}{\psi_k}\right)\sigma_Y + \sigma_{\xi,k}\right)\sigma_q,\\
\hat{\omega}^Y_k &= \frac{\mu_{Y,k} - \frac{1}{2}\sigma_Y^2 + \left(1-\frac{1}{\psi_k}\right)\sigma_Y^2}{\lambda_x}.
\end{aligned}
$$

To understand the intuition behind the prices of the spanning assets, it is useful to define the yield of a spanning asset via:

$$y^{\phi,Y}_{n,j,T-t} = -\frac{1}{T-t}\ln\phi^Y_{n,j,t,T}.$$

Hence,

$$y^{\phi,Y}_{n,1,T-t} = -\frac{1}{T-t}\ln v^{Y,hom}_{t,u}(k_1,b_1) - a_1\left[\hat{\mu}^Y_{q,1} + \frac{1}{2}a_1\sigma_q^2 + b_1\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x(T-t)}\sigma_q\sigma_Y\right] \tag{63}$$

$$-\frac{1}{T-t}\ln\Phi\left(\frac{B^Y_1(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right).$$

First, note that the yield in a homogeneous agent economy in which all agents are of Type 1 is

$$-\frac{1}{T-t}\ln v^{Y,hom}_{t,u}\left(k_1, -\left(\gamma_1 - \frac{1}{\psi_1}\right)\right).$$

We can then see that the first term in (63), $-\frac{1}{T-t}\ln v^{Y,hom}_{t,u}(k_1,b_1)$, is the expression for the yield in a homogeneous agent economy in which all agents are of type 1, with one change: the difference between the relative-risk aversion and the inverse of the sensitivity of the risk-free rate to consumption growth in the homogeneous agent economy is replaced by its weighted average, that is, $-b_1$, where b_1 is defined in (62). Given that the spanning asset is priced using the Hölder mean of the state-price densities of single agent economies, with weights $\frac{n}{\gamma_2}$ and $1-\frac{n}{\gamma_2}$, it is natural that the same weights appear in (62). The second term in (63) adjusts for the impact of heterogeneity on the distribution of consumption. The final term stems from $\Phi\left(\frac{B^Y_1(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right)$, the risk-adjusted probability of the underlying cash flow for the spanning asset being paid. An analogous interpretation holds for $y^{\phi,Y}_{n,2,T-t}$.

4.2 The equity risk premium and volatility of stock-market returns

The price of the stock, which can be interpreted as the market portfolio, and its cumulative return, R_t, which consists of capital gains plus dividends, is described by the process:

$$\frac{dP^Y_t + Y_t dt}{P^Y_t} = dR_t = \mu_{R,t}dt + \sigma_{R,t}dZ_t.$$

From Equation (25), we see that once we have the value of the spanning assets, one can obtain the price of equity by integrating the value of these spanning assets with respect to their maturities. The risk premium on equity, which pays Y_t in perpetuity, is given by the standard asset pricing equation:

$$E_t\left[\frac{dP^Y_t + Y_t dt}{P^Y_t} - r_t dt\right] = -E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP^Y_t}{P^Y_t}\right]. \tag{64}$$

Applying Ito's lemma to $P^Y_t = Y_t p^Y_t$ and using Equation (64) leads to the following proposition.

Proposition 9 (Volatility of stock market returns and equity risk premium). The volatility of stock market returns, $\sigma^Y_{R,t}$, is

$$\sigma^Y_{R,t} = \sigma_Y + \frac{1}{p^Y_t}\left(\frac{\partial p^Y_t}{\partial \omega_t}\sigma_Y + \frac{\partial p^Y_t}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_1,t}\right); \tag{65}$$

the risk premium on equity is

$$\mu^Y_{R,t} - r_t = \theta_t \sigma^Y_{R,t} = \left(\mathbf{R}_t \sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}\right]\right)\sigma^Y_{R,t}; \tag{66}$$

and, Agent k's perception of the risk premium is given by

$$\mu^Y_{R,k,t} - r_t = \left(\mathbf{R}_t \sigma_Y + \left[\frac{\mu_{Y,k} - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}\right]\right)\sigma^Y_{R,t}. \tag{67}$$

In a model with a single representative investor, stock return volatility, $\sigma_{R,t}$, is equal to fundamental volatility, σ_Y. From (65) we see that in a model with heterogeneous investors, stock market return volatility is the sum of fundamental volatility, σ_Y, and *excess volatility*, which is given by the second term: $\frac{1}{p^Y_t}(\frac{\partial p^Y_t}{\partial \omega_t}\sigma_Y + \frac{\partial p^Y_t}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_1,t})$. This second term depends on fluctuations in the price-dividend ratio. The excess-volatility term can be interpreted as the elasticity of the price-dividend ratio with respect to habit multiplied by the volatility of habit, plus the elasticity of the price-dividend ratio with respect to the consumption share multiplied by the volatility of the consumption share. When demand for precautionary savings is not too large, the price-dividend ratio is monotonic and countercyclical, and so excess volatility is positive, as in the data.

We now discuss the equity risk premium. From Proposition 9, we see that although agents agree on conditional stock return volatility, they may disagree on the conditional risk premium; that is, each agent will have her own perception of the equity risk premium, which is given by the expression in Equation (67). The central planner's view of the conditional risk premium, which is given in Equation (66), is the product of the market price of risk, θ_t, and the volatility of stock market returns, $\sigma^Y_{R,t}$. The risk premium will be high when: (1) in aggregate, agents are pessimistic, $\boldsymbol{\mu}_{Y,t} < \mu_Y$, (2) the aggregate risk aversion in the economy, $\mathbf{R}_t$, is high, and (3) stock return volatility, $\sigma^Y_{R,t}$, is high.[23]

Heterogeneity in risk aversion leads to cyclicality in the price of risk and stock market return volatility, which feeds into the risk premium, whereas heterogeneity in ψ or the rate of time preference has no such impact. In the data, the price of risk, stock market return volatility, and the conditional risk premium are all countercyclical, which can be generated by heterogeneity in risk aversion.

[23] Note that if stock return volatility, $\sigma^Y_{R,t}$, is higher than fundamental volatility, σ_Y, the risk premium can be higher than in either of the two homogeneous agent economies.

5. General Affine Model

In this section, we show how to price financial assets when the logarithm of aggregate endowment and agents' beliefs follow general affine processes, instead of the particular processes assumed in Equations (36) and (38).

We start by recalling that an *affine process* V_t with some state space $D \subset \mathbb{R}^d$ is defined as a Markov process whose conditional characteristic function is of the form, for any $a \in \mathbb{R}^d$,

$$E_t[e^{ia\cdot V_u}] = e^{\chi_1(t,u,z)+\chi_2(t,u,z)\cdot V_t},$$

for some coefficients $\chi_1(t,u,z)$ and $\chi_2(t,u,z)$. We also observe that if V_t is *analytic affine*, then $E_t[e^{ia\cdot V_u}]$, where each component of a is in $\mathbb{C}$, is a holomorphic function whose restriction to the real numbers is real valued. The class of analytic affine processes includes the class of affine jump diffusions in Duffie, Pan, and Singleton (2000).

Suppose $y = \ln Y$ is given by an analytic affine process with drift and diffusion driven by analytic affine processes. It then follows from (12) that

$$\hat{\pi}_{k,t} = e^{\varsigma_{k,t}},$$

where $\varsigma_{k,t}$ is an analytic affine process, given by

$$\varsigma_{k,t} = \ln(\lambda_{k,0}\xi_{k,t}) - \beta_k t - \left(\gamma_k - \frac{1}{\psi_k}\right)\omega_t - \frac{1}{\psi_k} y_t.$$

Consider a cash flow process, $X_t = e^{x_t}$, where x_t is analytic affine. The moment generating function

$$\mathcal{M}(\varsigma_{1,t}, \varsigma_{2,t}, x_t, a, b, c) = E_t[e^{a\varsigma_{1,T}+b\varsigma_{2,T}+cx_T}]$$

can be evaluated explicitly, because $\varsigma_{1,T}$, $\varsigma_{2,T}$, and x_T are jointly affine. The following proposition shows how one can obtain asset prices using the moment generating function, when the logarithm of aggregate endowment and agents' beliefs follow general analytic affine processes.

Proposition 10 (Values of $\phi^X_{n,1,t,u}$ and $\phi^X_{n,2,t,u}$ for general affine processes). If the logarithm of aggregate endowment and agents' beliefs follow general analytic affine processes, then the values of $\phi^X_{n,1,t,u}$ and $\phi^X_{n,2,t,u}$ that pay X_u at u whenever $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R$ and $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R$, respectively, are:

$$\begin{aligned}\phi_{n,1,t,u} &= e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t} \\ &\quad\times\left[\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\rho z}\frac{1}{z}\mathrm{Im}\left[\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)\right]dz\right. \\ &\quad\left.+\frac{1}{2}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,1-\frac{n}{\gamma_1},\frac{n}{\gamma_2},1\right)\right], \qquad (68)\end{aligned}$$

and

$$\phi_{n,2,t,u} = e^{-\left[\frac{n}{\gamma_1}\varsigma_{1,t}+\left(1-\frac{n}{\gamma_2}\right)\varsigma_{2,t}\right]-x_t}$$
$$\times\left[-\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\rho z}\frac{1}{z}\mathrm{Im}\left[\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\frac{n}{\gamma_1}+iz,1-\frac{n}{\gamma_2}-iz,1\right)\right]dz\right.$$
$$\left.+\frac{1}{2}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\frac{n}{\gamma_1},1-\frac{n}{\gamma_2},1\right)\right]. \quad (69)$$

6. Conclusion

In this paper, we study an endowment economy in which there are two types of agents, each with "catching up with the Joneses" utility. The two agents are heterogeneous with respect to their preference parameters for the subjective rate of time preference, relative-risk aversion, and sensitivity to habit, and also with respect to their beliefs.

Our main contribution is solving in closed form for the equilibrium in this economy and identifying the optimal consumption-sharing rule, without restricting the risk aversions of the two agents to particular values. We also identify the state price density, market price of risk, the locally risk-free interest rate the stock price, the equity market risk premium, the volatility of stock returns, and the term structure of interest rates. We derive the condition for the model to be stationary, in the sense that both types of agents survive in the long run. We then analyze how heterogeneity in preferences and beliefs affects the properties of asset returns.

We find that even when the aggregate belief is correct, heterogeneity in beliefs makes it easier to match the empirical properties of long-run asset returns. The aggregate risk aversion in the heterogeneous-agent economy is countercyclical, and consequently, the model is consistent also with dynamic properties of asset prices; for example, when aggregate consumption falls, expected stock returns, stock-return volatility, and the market price of risk rise, and price-dividend ratios decline.

Appendix: Lagrange's Theorem and Proofs for Propositions and Corollaries

We begin by stating a number of definitions and theorems from complex analysis that are used to derive results in the paper. In particular, the insight from Lagrange that is central to the analysis in the paper is given in Theorem A2.

Definition A1. If U is an open subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$ is a complex function on U, we say that f is *complex differentiable* at a point z_0 of U if the limit

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. The limit here is taken over all sequences of complex numbers approaching z_0, and for all such sequences the difference quotient has to approach the same number $f'(z_0)$.

Definition A2. If f is complex differentiable at every point z_0 in U, we say that f is *holomorphic* on U. We say that f is holomorphic at the point z_0 if it is holomorphic on some neighborhood of z_0. We say that f is holomorphic on some nonopen set A if it is holomorphic in an open set containing A.

Definition A3. A function f is *complex analytic* on an open set D in the complex plane if for any z_0 in D one can write

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

in which the coefficients a_0, a_1, ... are complex numbers and the series is convergent for z in a neighborhood of z_0.

Theorem A1 (Complex analytic). A function f is complex analytic on an open set D in the complex plane if and only if it is holomorphic in D.

We are now ready to state the theorem that allows us to find closed-form series expansions for the sharing rule and complex analytic functions of the sharing rule.

Theorem A2 (Lagrange). Suppose the dependence between the variables w and z is implicitly defined by an equation of the form

$$w=f(z),$$

where f is complex analytic in a neighborhood of 0 and $f'(0)\neq 0$. Then for any function g, which is complex analytic in a neighborhood of 0,

$$g(z)=g(0)+\sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{dx^{n-1}}g'(x)[\varphi(x)^n]\right]_{x=0}, \tag{A1}$$

where $\varphi(z)=\frac{z}{f(z)}$.

Note that the above theorem does not provide a radius of convergence for the series in Equation (A1). Although the original proof of Theorem A2 attributed to Lagrange is not very straightforward, a relatively easier proof can be obtained by using Cauchy's integral formula.

1. Proof of Proposition 1: Consumption-sharing rule

Equation (13) is equivalent to

$$A_t(1-\nu_{1,t})^{\eta}=\nu_{1,t},$$

which implicitly defines $\nu_{1,t}$ in terms of A_t. To solve explicitly for $\nu_{1,t}$, we apply Theorem A2, expanding around the point $\nu_{1,t}=0$, with

$$f(z)=z(1-z)^{-\eta}, \tag{A2}$$

$$\varphi(z)=(1-z)^{\eta}, \tag{A3}$$

$$g(z)=z,$$

after showing that f is complex analytic in some neighborhood of 0. We know from the binomial series expansion that for $z\in\mathbb{C}$, such that $|z|<1$,

$$(1-z)^{-\eta}=\sum_{n=0}^{\infty}\binom{-\eta}{k}(-)^n z^n,$$

where $\binom{-\eta}{k}=\Pi_{j=1}^{k}\frac{-\eta-k+j}{j}$ is the generalized binomial coefficient. Therefore, $(1-z)^{-\eta}$ is complex analytic in the open ball $\{z\in\mathbb{C}:|z|<1\}$. Because z is complex analytic for all $z\in\mathbb{C}$, it follows that

f as defined in (A2) is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. It therefore follows from Theorem A2 that

$$\nu_{1,t} = \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{\eta n}\right]_{x=0}.$$

Because

$$\frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{n\eta}\right] = (-)^{n-1} \eta n(\eta n-1)(\eta n-2)\ldots(\eta n-(n-2))(1-x)^{\eta n-(n-1)},$$

it follows that

$$\nu_{1,t} = -\sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1},$$

$$\nu_{2,t} = 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1}. \tag{A4}$$

We shall now determine the radius of convergence of the above series. From d'Alembert's ratio test, it follows that the above series converge absolutely for all $A \in \mathbb{C}$ s.t. $|A| < \overline{R}$, where

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \left| \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}} \right|.$$

We wish to evaluate the above limit for all $\eta \in \mathbb{R}$ such that $\eta > 1$. Hence, $\binom{\eta n}{n-1}$ and $\binom{\eta(n+1)}{n}$ are positive and real, and so

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}}.$$

We note that the generalized binomial coefficient, $\binom{z}{k} = \Pi_{j=1}^{k} \frac{z-k+j}{j}$, can be written as

$$\binom{z}{k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\Gamma(k+1)}, \tag{A5}$$

where $\Gamma(z)$ is the Gamma function, which for $\Re(z) > 0$ (where $\Re(z)$ denotes the real part of z), has the integral representation,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

The Euler beta function, $B(x, y)$, defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

can be written in terms of the gamma function as follows,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{A6}$$

Together with (A5), the above expression implies that the generalized binomial coefficient is given by

$$\binom{z}{k} = \frac{1}{(z+1)B(z-k+1, k+1)}. \tag{A7}$$

Hence,

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{B((\eta-1)(n+1), n+1)}{B((\eta-1)n, n)}.$$

To evaluate the above limit, we start by recalling Stirling's series for the gamma function

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left(1 + O\left(\frac{1}{z}\right)\right), \tag{A8}$$

which together with (A6) implies that

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\dfrac{((\eta-1)(n+1))^{(\eta-1)(n+1)-\frac{1}{2}}(n+1)^{(n+1)-\frac{1}{2}}}{((\eta-1)(n+1)+(n+1))^{((\eta-1)(n+1)+(n+1))-\frac{1}{2}}}}{\dfrac{((\eta-1)n)^{((\eta-1)n)-\frac{1}{2}} n^{n-\frac{1}{2}}}{(((\eta-1)n)+n)^{(((\eta-1)n)+n)-\frac{1}{2}}}}.$$

Simplifying the above expression gives

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{(\eta-1)^{\eta-1}}{\eta^\eta} \sqrt{\frac{n}{n+1}}$$
$$= \frac{(\eta-1)^{\eta-1}}{\eta^\eta}.$$

Because A_t is a geometric Brownian motion, it is positive and real. Hence, the right-hand side of (A4) is absolutely convergent for $A_t < \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$.

We now derive a series expansion for $\nu_{2,t}$ in terms of A_t, which is absolutely convergent for $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$. We start by rearranging (13) to obtain

$$\nu_{2,t} = A_t^{-1/\eta}(1-\nu_{2,t})^{1/\eta}.$$

To find $\nu_{2,t}$, we apply Theorem A2, expanding around the point $\nu_{2,t}=0$, with f, φ and g, defined as below

$$f(z) = z(1-z)^{-1/\eta}, \tag{A9}$$

$$\varphi(z) = (1-z)^{1/\eta}, \tag{A10}$$

$$g(z) = z.$$

We can show that our newly defined f is complex analytic in the open ball, $\{z \in \mathbb{C} : |z| < 1\}$, in the same way as for (A2). Hence, Theorem A2 implies that

$$\nu_{2,t} = \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{n/\eta}\right]_{x=0}.$$

Because

$$\frac{d^{n-1}}{dx^{n-1}}\left[(1-x)^{n/\eta}\right] = (-)^{n-1} \frac{n}{\eta}\left(\frac{n}{\eta}-1\right)\left(\frac{n}{\eta}-2\right)\dots\left(\frac{n}{\eta}-(n-2)\right)(1-x)^{\frac{n}{\eta}-(n-1)},$$

it follows that

$$\nu_{2,t} = -\sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \binom{\frac{n}{\eta}}{n-1} = \sum_{n=1}^{\infty} \frac{(-)^{n-1}\left(A_t^{-\frac{1}{\eta}}\right)^n}{n} \binom{\frac{n}{\eta}}{n-1}.$$

By comparing the above expression with (1.), we can see that (1.) is absolutely convergent if $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}^{\frac{1}{\eta}}}$, that is, if $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$. To summarize, we have

$$\nu_{2,t} = \begin{cases} -\sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \binom{\frac{n}{\eta}}{n-1} & , A_t > \overline{R}, \\ 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta}{n-1} & , A_t < \overline{R}, \end{cases}$$

where $\overline{R}$ is given in (1.). Using (2.1) we can write the expressions for the sharing rule as (14).

2. Proof of Proposition 2: State-price density

The equilibrium state price density is given by (46). To find a closed-form expression for the equilibrium state-price density, we find series expansions for $\nu_{k,t}^{-\gamma_k}$, $k\in\{1,2\}$. To find a series expansion for $\nu_{2,t}^{-\gamma_2}$, note that

$$\nu_{2,t}^{-\gamma_2}=(1-\nu_{1,t})^{-\gamma_2},$$

and use Theorem A2 to expand around the point $\nu_{1,t}=0$. To do this, we define

$$g(z)=(1-z)^{-\gamma_2},$$

which is complex analytic in the open ball $\{z\in\mathbb{C}:|z|<1\}$. Hence, with f and φ defined as in (A2) and (A3), respectively, Theorem A2 implies that

$$\begin{aligned} g(\nu_{1,t})&=(1-\nu_{1,t})^{-\gamma_2} \\ &=g(0)+\sum_{n=1}^{\infty}\frac{A_t^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[g'(x)\varphi(x)^n\right]_{x=0} \\ &=1+\sum_{n=1}^{\infty}\frac{A_t^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[\gamma_2(1-x)^{n\eta-\gamma_2-1}\right]_{x=0}. \end{aligned}$$

Because

$$\begin{aligned} &\frac{d^{n-1}}{dx^{n-1}}\gamma_2(1-x)^{n\eta-\gamma_2-1} \\ &=\gamma_2(-)^{n-1}(n\eta-\gamma_2-1)(n\eta-\gamma_2-2)\dots(n\eta-\gamma_2-(n-1))(1-x)^{n\eta-\gamma_2-(n-1)}, \end{aligned}$$

it follows that

$$\nu_{2,t}^{-\gamma_2}=1-\gamma_2\sum_{n=1}^{\infty}\frac{(-A_t)^n}{n}\binom{n\eta-\gamma_2-1}{n-1}. \tag{A11}$$

D'Alembert's ratio test implies that the above series converges absolutely for all $A\in\mathbb{C}$ such that $|A|<\overline{R}$, where

$$\overline{R}=\lim_{n\to\infty}\frac{n+1}{n}\frac{\binom{\eta n-\gamma_2-1}{n-1}}{\binom{\eta(n+1)-\gamma_2-1}{n}}.$$

Using (A7), we rewrite the above expression as

$$\overline{R}=\lim_{n\to\infty}\frac{n+1}{n}\frac{\eta(n+1)-\gamma_2}{\eta n-\gamma_2}\frac{B((\eta-1)(n+1)-\gamma_2-1,n+1)}{B((\eta-1)n-\gamma_2-1,n)}.$$

Hence, using (A6) and (A8), we obtain

$$\overline{R}=\lim_{n\to\infty}\frac{n+1}{n}\frac{\eta(n+1)-\gamma_2}{\eta n-\gamma_2}\frac{\dfrac{[(\eta-1)(n+1)-(1+\gamma_2)]^{(\eta-1)(n+1)-(1+\gamma_2)-1/2}(n+1)^{n+1-1/2}}{[\eta(n+1)-(1+\gamma_2)]^{\eta(n+1)-(1+\gamma_2)-1/2}}}{\dfrac{[(\eta-1)n-(1+\gamma_2)]^{(\eta-1)n-(1+\gamma_2)-1/2}n^{n-1/2}}{[\eta n-(1+\gamma_2)]^{\eta n-(1+\gamma_2)-1/2}}}.$$

Simplifying the above expression gives

$$\overline{R}=\frac{(\eta-1)^{(\eta-1)}}{\eta^{\eta}}.$$

Because A_t is a geometric Brownian motion, A_t is real and positive, and so the right-hand side of (A11) is absolutely convergent if $A_t < \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta} = \overline{R}$. Hence,

$$\nu_{2,t}^{-\gamma_2} = 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \begin{pmatrix} n\eta - \gamma_2 - 1 \\ n-1 \end{pmatrix}, A_t < \overline{R}.$$

Using (12) and (16), we can rewrite the above expression as

$$\nu_{2,t}^{-\gamma_2} = \sum_{n=0}^{\infty} a_{n,2}^{\pi} \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}}, \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\gamma_2 - \gamma_1},$$

where $a_{n,2}^{\pi}$ is defined in (19). Therefore, the equilibrium state-price density is given by

$$\pi_t = \sum_{n=0}^{\infty} a_{n,2}^{\pi} \hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}, \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\gamma_2 - \gamma_1}. \tag{A12}$$

To find an expression for the state-price density when $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$, we find a series expansion for $\nu_{1,t}^{-\gamma_1}$, which is absolutely convergent for $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$. Note that

$$\nu_{1,t}^{-\gamma_1} = (1 - \nu_{2,t})^{-\gamma_1},$$

and use Theorem A2 to expand around the point $\nu_{2,t} = 0$. To do this, we define

$$g(z) = (1-z)^{-\gamma_1},$$

which is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. Hence, with f and φ defined as in (A9) and (A10), respectively, Theorem A2 implies that

$$\begin{aligned} g(\nu_{2,t}) &= (1-\nu_{2,t})^{-\gamma_1} \\ &= g(0) + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[g'(x)\varphi(x)^n\right]_{x=0} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[\gamma_1 (1-x)^{\frac{n}{\eta} - \gamma_1 - 1}\right]_{x=0}. \end{aligned}$$

Because,

$$\begin{aligned} &\frac{d^{n-1}}{dx^{n-1}} \left[\gamma_1 (1-x)^{\frac{n}{\eta} - \gamma_1 - 1}\right] \\ &= \gamma_1 (-)^{n-1} \left(\frac{n}{\eta} - \gamma_1 - 1\right)\left(\frac{n}{\eta} - \gamma_1 - 2\right)\ldots\left(\frac{n}{\eta} - \gamma_1 - (n-1)\right)(1-x)^{\frac{n}{\eta} - \gamma_1 - (n-1)}, \end{aligned}$$

it follows that

$$\nu_{1,t}^{-\gamma_1} = 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{\left(-A_t^{-1/\eta}\right)^n}{n} \begin{pmatrix} \frac{n}{\eta} - \gamma_1 - 1 \\ n-1 \end{pmatrix}. \tag{A13}$$

By comparing the above expression with (A11), we can see that (A13) is absolutely convergent if $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}^{\frac{1}{\eta}}}$, that is, if $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta} = \overline{R}$. Thus,

$$\nu_{1,t}^{-\gamma_1} = 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{\left(-A_t^{-1/\eta}\right)^n}{n} \begin{pmatrix} \frac{n}{\eta} - \gamma_1 - 1 \\ n-1 \end{pmatrix}, A_t > \overline{R}.$$

Using (12) and (16), we can rewrite the above expression as

$$v_{1,t}^{-\gamma_1}=\sum_{n=0}^{\infty}a_{n,1}^{\pi}\left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}},\ \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}>\frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1},$$

where $a_{n,1}^{\pi}$ is defined in (18). Therefore, the equilibrium state-price density is given by

$$\pi_t=\sum_{n=0}^{\infty}a_{n,2}^{\pi}\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}},\ \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}>\frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1}. \tag{A14}$$

The expressions in (17) follow from (A12) and (A14).

3. Proof of Corollary 1: State-price density under identical risk aversion

First, we note that

$$\lim_{a\to 0}\left(\frac{\gamma+a}{\gamma}-1\right)^{a}=1.$$

Therefore, setting $\gamma_1=\gamma_2=\gamma$ implies that

$$\frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1}=1.$$

Also, note that after some tedious algebra, we can show that

$$\gamma\binom{n-\gamma-1}{n}\frac{(-)^{n+1}}{n}=\binom{\gamma}{n}.$$

Therefore, when $\gamma_1=\gamma_2=\gamma$, (18) and (19) reduce to (21).

The expression in (23) is obtained from (20) by using Newton's binomial theorem (for nonintegral powers). When γ is a natural number, the expression in (23) can be obtained also from (22) by using the binomial theorem for integral powers, and one could also obtain (23) directly from the first-order condition for consumption in (11).

4. Proof of Proposition 3: Price of dividend strip

Rather than considering a dividend strip based where the aggregate endowment, Y, is the dividend, we shall derive results for a more general dividend strip, which pays out X_u at date u, where the evolution of X is given by

$$\frac{dX_t}{X_t}=\mu_X dt+\sigma_X^{sys}dZ_t+\sigma_X^{id}dZ_t^{id},$$

where Z_t^{id} is a standard Brownian motion under $\mathbb{P}$, orthogonal to Z_t, and

$$\frac{\mu_{Y,k}-\mu_Y}{\sigma_Y}=\frac{\mu_{X,k}-\mu_X}{\sigma_X^{sys}}.$$

The date-t price of the dividend strip, which pays out X_u at date $u>t$, is denoted by $V_{t,u}^X$, where

$$V_{t,u}^X=v_{t,u}^X X_t,$$

and

$$v_{t,u}^X=E_t\int_t^{\infty}\left[\frac{\pi_u}{\pi_t}\frac{X_u}{X_t}dt\right]. \tag{A15}$$

We shall derive an expression for $v_{t,u}^X$, and then, to get the equations in the proposition giving the price of a dividend strip, where the endowment is the dividend, we will set $\mu_X=\mu_Y$, $\sigma_X^{sys}=\sigma_Y$, and $\sigma_X^{id}=0$.

To derive a closed-form expression for the price-dividend ratio in (A15), we use (17) to write the equilibrium state-price density as

$$\pi_t = \sum_{n=0}^{\infty} a^{\pi}_{n,1} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} 1_{\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R\right\}} + \sum_{n=0}^{\infty} a^{\pi}_{n,2} \hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}} 1_{\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R\right\}}.$$

Because the event $\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} = R\right\}$ is of measure zero, it follows from (A15) that

$$v^X_{t,u} = (\pi_t X_t)^{-1} j_{t,u}, \tag{A16}$$

where

$$j_{t,u} = E_t\left[\sum_{n=0}^{\infty} a^{\pi}_{n,1} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,u}^{\frac{n}{\gamma_2}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R\right\}} + \sum_{n=0}^{\infty} a^{\pi}_{n,2} \hat{\pi}_{1,u}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right\}}\right].$$

Because the two infinite series in the above expression stem from $\nu_{2,t}^{-\gamma_2}$ in (A11), and $\nu_{1,t}^{-\gamma_1}$ in (A13), which are complex analytic for $A \in \mathbb{C}$ such that $|A| < \overline{R}$, and $|A| > \overline{R}$, respectively, we can interchange the conditional expectation with the infinite sum to obtain

$$j_{t,u} = \sum_{n=0}^{\infty} a^{\pi}_{n,1} E_t\left[\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,u}^{\frac{n}{\gamma_2}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R\right\}}\right] + \sum_{n=0}^{\infty} a^{\pi}_{n,2} E_t\left[\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right\}}\right].$$

We now rewrite the above expression as follows:

$$j_{t,u} = \pi_t X_t \left(\sum_{n=0}^{\infty} \omega_{n,1,t} \phi^X_{n,1,t,u} + \sum_{n=0}^{\infty} \omega_{n,2,t} \phi^X_{n,2,t,u}\right), \tag{A17}$$

where $\omega_{n,1,t}$ and $\omega_{n,2,t}$ are given by

$$\omega_{n,1,t} = a^{\pi}_{n,1} \frac{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}{\pi_t}, n \in \mathbb{N}_0, \tag{A18}$$

$$\omega_{n,2,t} = a^{\pi}_{n,2} \frac{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}{\pi_t}, n \in \mathbb{N}_0, \tag{A19}$$

and $\phi^X_{n,1,t,u}$ and $\phi^X_{n,2,t,u}$ are given by

$$\phi^X_{n,1,t,u} = E_t\left[\frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}} \frac{X_u}{X_t} 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R\right\}}\right], n \in \mathbb{N}_0,$$

$$\phi^X_{n,2,t,u} = E_t\left[\frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}} \frac{X_u}{X_t} 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right\}}\right], n \in \mathbb{N}_0.$$

We now set $\mu_X = \mu_Y$, $\sigma_X^{sys} = \sigma_Y$, and $\sigma_X^{id} = 0$, and so Equation (26) follows from (A16) and (A17). We also note that (29) follows from (17), (A18), and (A19).

We now express the weights, $\omega_{n,1,t}$ and $\omega_{n,2,t}$, in terms of the consumption shares, $\nu_{1,t}$ and $\nu_{2,t}$. Equation (11) implies that for all $a \in \mathbb{R}$

$$\pi_t = \hat{\pi}_{1,t}^a \nu_{1,t}^{-a\gamma_1} \hat{\pi}_{2,t}^{1-a} \nu_{2,t}^{-(1-a)\gamma_2},$$

which implies that

$$\hat{\pi}_{1,t}^a \hat{\pi}_{2,t}^{1-a} = \pi_t \nu_{1,t}^{a\gamma_1} \nu_{2,t}^{(1-a)\gamma_2}.$$

Therefore, we can rewrite the weights $\omega_{n,1,t}$ and $\omega_{n,2,t}$, given in (A18) and (A19), as (27) and (28), respectively.

5. Proof of Corollary 2: Price of dividend strip with identical risk aversion

Again, rather than considering dividend strips, where the dividend is the aggregate endowment, we shall derive results for a more general dividend process, X, where the evolution of X is given by (A50). Then, to obtain the results in the proposition, we will set $\mu_X = \mu_Y$, $\sigma_X^{sys} = \sigma_Y$, and $\sigma_X^{id} = 0$.

By setting $\gamma_1 = \gamma_1 = \gamma$, (27) and (28) reduce to (32) and (33), respectively, and (30) and (31) reduce to

$$\phi_{n,1,t,u}^X = E_t \left[\frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,u}^{\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,t}^{\frac{n}{\gamma}}} \frac{X_u}{X_t} 1_{\left\{ \frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R \right\}} \right], n \in \mathbb{N}_0, \tag{A20}$$

$$\phi_{n,2,t,u}^X = E_t \left[\frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma}} \hat{\pi}_{2,u}^{1-\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma}}} \frac{X_u}{X_t} 1_{\left\{ \frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R \right\}} \right], n \in \mathbb{N}_0. \tag{A21}$$

When $\gamma \in \mathbb{N}$, (26) reduces to

$$v_{t,u}^X = \sum_{n=0}^{\gamma} \omega_{n,1,t} \phi_{n,1,t,u}^X + \sum_{n=0}^{\gamma} \omega_{n,2,t} \phi_{n,2,t,u}^X = \sum_{n=0}^{\gamma} \omega_{n,t} \left(\phi_{n,1,t,u}^X + \phi_{\gamma-n,2,t,u}^X \right),$$

where $\omega_{n,t}$ is given in (35). It follows from (A20) and (A21) that

$$\phi_{n,1,t}^X + \phi_{\gamma-n,2,t}^X = E_t \left[\frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,u}^{\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,t}^{\frac{n}{\gamma}}} \frac{X_u}{X_t} \right], n \in \mathbb{N}_0 \text{ and } n \leq \gamma.$$

The n'th weight in the sum is given by the expression in (35); observe that the weights sum to one, because

$$\sum_{n=0}^{\gamma} \binom{\gamma}{n} \left(\nu_{1,t}^{1-\frac{n}{\gamma}} \nu_{2,t}^{\frac{n}{\gamma}} \right)^{\gamma} = (\nu_{1,t} + \nu_{2,t})^{\gamma} = 1.$$

6. Proof of Proposition 4: Dynamics of the consumption-sharing rule

We can see the state variables of the economy by writing the consumption sharing rule as

$$e^{\Delta_t} \nu_{1,t}^{-\gamma_1} = \nu_{2,t}^{-\gamma_2}, \tag{A22}$$

where

$$\Delta_t = \ln \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} = q_t + d_\omega \omega_t,$$

and

$$q_t = \ln \frac{\lambda_{1,0}}{\lambda_{2,0}} + \mu_q t + \sigma_q Z_t,$$

$$\mu_q = (\beta_2 - \beta_1) + \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1} \right) \left(\mu_Y - \frac{1}{2}\sigma_Y^2 \right),$$

$$\sigma_q = \sigma_{\xi,1} - \sigma_{\xi,2} + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1} \right) \sigma_Y,$$

$$d_\omega = \left(\gamma_2 - \frac{1}{\psi_2} \right) - \left(\gamma_1 - \frac{1}{\psi_1} \right).$$

Hence, the state variables of the economy are q and ω.

We observe from (16) that the evolution of the ratio $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}$ will have a deterministic component and a stochastic component, where the stochastic component depends on the stochastic behavior of aggregate endowment and the differences in beliefs.

From Ito's lemma

$$d\nu_{1,t} = \frac{\partial \nu_{1,t}}{\partial \Delta_t} d\Delta_t + \frac{1}{2}\frac{\partial^2 \nu_{1,t}}{\partial \Delta_t^2}(d\Delta_t)^2.$$

Differentiating (A22) implicitly with respect Δ_t and solving for $\frac{\partial \nu_{1,t}}{\partial \Delta_t}$ gives

$$\frac{\partial \nu_{1,t}}{\partial \Delta_t} = \mathbf{R}_t \frac{\nu_{1,t}\nu_{2,t}}{\gamma_1\gamma_2}. \tag{A23}$$

Differentiating (A23) with respect to Δ_t gives

$$\begin{aligned}\frac{\partial^2 \nu_{1,t}}{\partial \Delta_t^2} &= \frac{\partial}{\partial \Delta_t}\left[\mathbf{R}_t \frac{\nu_{1,t}\nu_{2,t}}{\gamma_1\gamma_2}\right] \\ &= \frac{1}{\gamma_1\gamma_2}\frac{\partial}{\partial \Delta_t}\left[\mathbf{R}_t \nu_{1,t}\nu_{2,t}\right] \\ &= \frac{1}{\gamma_1\gamma_2}\left[\mathbf{R}_t \frac{\partial}{\partial \Delta_t}\left[\nu_{1,t}\nu_{2,t}\right] + \nu_{1,t}\nu_{2,t}\frac{\partial \mathbf{R}_t}{\partial \Delta_t}\right].\end{aligned}$$

We simplify the above expression by observing that

$$\frac{\partial \mathbf{R}_t}{\partial \Delta_t} = -\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\frac{\partial \nu_{1,t}}{\partial \Delta_t}.$$

and using (A23) to obtain

$$\frac{\partial^2 \nu_{1,t}}{\partial \Delta_t^2} = \nu_{1,t}\nu_{2,t}\left(\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\right)^2\left[(\nu_{2,t} - \nu_{1,t}) - \nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right].$$

Hence,

$$\begin{aligned}d\nu_{1,t} &= \frac{\partial \nu_{1,t}}{\partial \Delta_t} d\Delta_t + \frac{1}{2}\frac{\partial^2 \nu_{1,t}}{\partial \Delta_t^2}(d\Delta_t)^2 \\ &= \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Bigg\{\mu_q dt + \sigma_q dZ_t + d_\omega[\lambda_x(\overline{\omega} - \omega_t)]dt + d_\omega \sigma_Y dZ_t \\ &\quad + \frac{1}{2}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left[\nu_{2,t} - \nu_{1,t} - \nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right](\sigma_q + d_\omega\sigma_Y)^2 dt\Bigg\}.\end{aligned}$$

Now observe that

$$\begin{aligned}\sigma_q + d_\omega \sigma_Y &= \sigma_{\xi,1} - \sigma_{\xi,2} + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\sigma_Y + \left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\sigma_Y \\ &= \sigma_{\xi,1} - \sigma_{\xi,2} + (\gamma_2 - \gamma_1)\sigma_Y.\end{aligned}$$

Therefore,

$$\begin{aligned}d\nu_{1,t} &= \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Bigg\{\mu_q + d_\omega[\lambda_x(\overline{\omega} - \omega_t)] \\ &\quad + \frac{1}{2}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left[\nu_{2,t} - \nu_{1,t} - \nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right][\sigma_{\xi,1} - \sigma_{\xi,2} + (\gamma_2 - \gamma_1)\sigma_Y]^2\Bigg\}dt \\ &\quad + \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}[\sigma_{\xi,1} - \sigma_{\xi,2} + (\gamma_2 - \gamma_1)\sigma_Y]dZ_t.\end{aligned}$$

Now observe that

$$\begin{aligned}
&\mu_q + d_\omega[\lambda_x(\overline{\omega} - \omega_t)] \\
&= (\beta_2 - \beta_1) + \frac{1}{2}(\sigma^2_{\xi,2} - \sigma^2_{\xi,1}) + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\left(\mu_Y - \frac{1}{2}\sigma^2_Y\right) \\
&\quad \times \left[\left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\right]\left(\mu_Y - \frac{1}{2}\sigma^2_Y\right) - \left[\left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\right]\lambda_x \omega_t \\
&= \beta_2 - \beta_1 + \frac{1}{2}(\sigma^2_{\xi,2} - \sigma^2_{\xi,1}) + (\gamma_2 - \gamma_1)\left(\mu_Y - \frac{1}{2}\sigma^2_Y\right) - \left[\left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\right]\lambda_x \omega_t
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left[\nu_{2,t} - \nu_{1,t} - \nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right] &= \left(\frac{\mathbf{R}_t \nu_{2,t}}{\gamma_2}\right)\frac{1}{\gamma_1} - \left(\frac{\mathbf{R}_t \nu_{1,t}}{\gamma_1}\right)\frac{1}{\gamma_2} - \frac{\mathbf{R}_t \nu_{1,t}}{\gamma_1}\frac{\mathbf{R}_t \nu_{2,t}}{\gamma_2}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) \\
&= \frac{w_{2,t}}{\gamma_1} - \frac{w_{1,t}}{\gamma_2} - w_{1,t}w_{2,t}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) = \frac{w^2_{2,t}}{\gamma_1} - \frac{w^2_{1,t}}{\gamma_2} \\
&= \frac{\gamma_2 w^2_{2,t} - \gamma_1 w^2_{1,t}}{\gamma_1\gamma_2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
d\nu_{1,t} &= \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2} \\
&\quad \times \Bigg\{(\beta_2 - \beta_1) + \frac{1}{2}(\sigma^2_{\xi,2} - \sigma^2_{\xi,1}) + (\gamma_2 - \gamma_1)\left(\mu_Y - \frac{1}{2}\sigma^2_Y\right) - \left[\left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\right]\lambda_x \omega_t \\
&\quad + \frac{1}{2}\frac{\gamma_2 w^2_{2,t} - \gamma_1 w^2_{1,t}}{\gamma_1\gamma_2}\Big[(\sigma_{\xi,1} - \sigma_{\xi,2}) + (\gamma_2 - \gamma_1)\sigma_Y\Big]^2\Bigg\}dt \\
&\quad + \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Big[(\sigma_{\xi,1} - \sigma_{\xi,2}) + (\gamma_2 - \gamma_1)\sigma_Y\Big]dZ_t \\
&= \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Bigg\{(\beta_2 - \beta_1) + \frac{1}{2}(\sigma^2_{\xi,2} - \sigma^2_{\xi,1}) \\
&\quad + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\left(\mu_Y - \frac{1}{2}\sigma^2_Y\right) + \left[\left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right)\right]\lambda_x(\bar{\omega} - \omega_t) \\
&\quad + \frac{1}{2}\frac{\gamma_2 w^2_{2,t} - \gamma_1 w^2_{1,t}}{\gamma_1\gamma_2}\Big[(\sigma_{\xi,1} - \sigma_{\xi,2}) + (\gamma_2 - \gamma_1)\sigma_Y\Big]^2\Bigg\}dt \\
&\quad + \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\Big[(\sigma_{\xi,1} - \sigma_{\xi,2}) + (\gamma_2 - \gamma_1)\sigma_Y\Big]dZ_t
\end{aligned}$$

$$= \nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left\{\beta_2-\beta_1+\left(\frac{\frac{\mu_{Y,1}+\mu_{Y,2}}{2}-\mu_Y}{\sigma_Y}\right)\left(\frac{\mu_{Y,2}-\mu_{Y,1}}{\sigma_Y}\right)\right.$$

$$+\left(\frac{1}{\psi_2}-\frac{1}{\psi_1}\right)\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right)+\left[\left(\gamma_2-\frac{1}{\psi_2}\right)-\left(\gamma_1-\frac{1}{\psi_1}\right)\right]\lambda_x(\bar{\omega}-\omega_t)$$

$$\left.+\frac{1}{2}\frac{\gamma_2 w_{2,t}^2-\gamma_1 w_{1,t}^2}{\gamma_1\gamma_2}\left[\left(\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}\right)^2+2(\mu_{Y,1}-\mu_{Y,2})(\gamma_2-\gamma_1)+(\gamma_2-\gamma_1)^2\sigma_Y^2\right]\right\}dt$$

$$+\nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left[\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}+(\gamma_2-\gamma_1)\sigma_Y\right]dZ_t.$$

7. Proof of Proposition 5: Risk-free rate

We know that the equilibrium state price density, π, is given by

$$\pi_t=\hat{\pi}_{1,t}\nu_{1,t}^{-\gamma_1}=\hat{\pi}_{2,t}\nu_{2,t}^{-\gamma_2}.$$

Hence,

$$\ln\pi_t=\ln\hat{\pi}_{1,t}-\gamma_1\ln\nu_{1,t}.$$

In the homogeneous agent economy, in which all agents are of Type k, we derive the risk-free rate, $\hat{r}_{k,t}$, and the market price of risk, $\hat{\theta}_{k,t}$. We start from the well-known result that

$$\frac{d\hat{\pi}_{k,t}}{\hat{\pi}_{k,t}}=-\hat{r}_{k,t}dt-\hat{\theta}_{k,t}dZ_t.$$

Given our assumptions for the aggregate endowment, we can apply Ito's lemma to the last expression in (12)

$$d\ln\hat{\pi}_{k,t}=-\frac{1}{2}\sigma_{\xi,k}^2dt+\sigma_{\xi,k}dZ_t-\beta_k dt-\left(\gamma_k-\frac{1}{\psi_k}\right)d\omega_t+\frac{1}{\psi_k}dy_t$$

$$=-\frac{1}{2}\sigma_{\xi,k}^2dt-\beta_k dt-\left(\gamma_k-\frac{1}{\psi_k}\right)[\lambda_x(\bar{\omega}-\omega)dt+\sigma_Y dZ_t]+\frac{1}{\psi_k}\left[\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right)dt+\sigma_Y dZ_t\right].$$

Therefore,

$$\hat{\theta}_k=-\sigma_{\xi,k}+\left(\gamma_k-\frac{1}{\psi_k}\right)\sigma_Y+\frac{1}{\psi_k}\sigma_Y=\gamma_k\sigma_Y+\frac{\mu_Y-\mu_{Y,k}}{\sigma_Y},$$

and

$$r_{k,t}+\frac{1}{2}\hat{\theta}_{k,t}^2=r_{k,t}+\frac{1}{2}(\gamma_k\sigma_Y-\sigma_{\xi,k})=\frac{1}{2}\sigma_{\xi,k}^2+\beta_k+\left(\gamma_k-\frac{1}{\psi_k}\right)\lambda_x(\bar{\omega}-\omega_t)+\frac{1}{\psi_k}\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right).$$

Hence,

$$\hat{r}_{k,t}+\frac{1}{2}\left(\sigma_{\xi,k}^2-2\gamma_k\sigma_Y\sigma_{\xi,k}+\gamma_k^2\sigma_{\xi,k}^2\right)=\frac{1}{2}\sigma_{\xi,k}^2+\beta_k+\left(\gamma_k-\frac{1}{\psi_k}\right)\lambda_x(\bar{\omega}-\omega_t)+\frac{1}{\psi_k}\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right)$$

$$\hat{r}_{k,t}-\gamma_k(\mu_{Y,k}-\mu_Y)+\frac{1}{2}\gamma_k^2\sigma_Y^2=\beta_k+\left(\gamma_k-\frac{1}{\psi_k}\right)\lambda_x(\bar{\omega}-\omega_t)+\frac{1}{\psi_k}\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right),$$

and so

$$\hat{r}_{k,t}=\beta_k+\frac{1}{\psi_k}\mu_{Y,k}+\lambda_x(\bar{\omega}_k-\omega_t)\left(\gamma_k-\frac{1}{\psi_k}\right)-\frac{1}{2}\left[\gamma_k(1+\gamma_k)-\left(\gamma_k-\frac{1}{\psi_k}\right)\right]\sigma_Y^2, \tag{A24}$$

where

$$\bar{\omega}_k=\frac{\mu_{Y,k}-\frac{1}{2}\sigma_Y^2}{\lambda_x}.$$

Applying Ito's lemma to $\ln\pi_t$ gives

$$d\ln\pi_t=d\ln\hat{\pi}_{1,t}-\gamma_1 d\ln\nu_{1,t}=-\left(\hat{r}_{1,t}+\frac{1}{2}\hat{\theta}_{1,t}^2\right)dt-\hat{\theta}_{1,t}dZ_t-\gamma_1\left[\left(\mu_{\nu_1,t}-\frac{1}{2}\sigma_{\nu_1,t}^2\right)dt+\sigma_{\nu_1,t}dZ_t\right]$$

$$=-\left[\hat{r}_{1,t}+\frac{1}{2}\hat{\theta}_{1,t}^2+\gamma_1\left(\mu_{\nu_1,t}-\frac{1}{2}\sigma_{\nu_1,t}^2\right)\right]dt-(\hat{\theta}_{1,t}+\gamma_1\sigma_{\nu_1,t})dZ_t.$$

Therefore

$$\begin{aligned}\theta_t&=\hat{\theta}_{1,t}+\gamma_1\sigma_{\nu_1,t}\\&=\frac{\mu_Y-\mu_{Y,1}}{\sigma_Y}+\gamma_1\sigma_Y+\frac{\nu_{1,t}\mathbf{R}_t}{\gamma_2}\left[\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}+(\gamma_2-\gamma_1)\sigma_Y\right]\\&=\frac{\mu_Y-\left[\left(1-\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\right)\mu_{Y,1}+\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\mu_{Y,2}\right]}{\sigma_Y}+\left[\gamma_1\left(1-\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\right)+\gamma_2\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\right]\sigma_Y\\&=\frac{\mu_Y-\left(\frac{\nu_{1,t}\mathbf{R}_t}{\gamma_1}\mu_{Y,1}+\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\mu_{Y,2}\right)}{\sigma_Y}+\left(\gamma_1\frac{\nu_{1,t}\mathbf{R}_t}{\gamma_1}+\gamma_2\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\right)\sigma_Y\\&=\frac{\mu_Y-\left(\frac{\nu_{1,t}\mathbf{R}_t}{\gamma_1}\mu_{Y,1}+\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}\mu_{Y,2}\right)}{\sigma_Y}+\mathbf{R}_t\sigma_Y,\end{aligned}$$

where we have used the fact that

$$\frac{\nu_{1,t}\mathbf{R}_t}{\gamma_1}=1-\frac{\nu_{2,t}\mathbf{R}_t}{\gamma_2}.$$

Therefore,

$$\theta_t=\frac{\mu_Y-\boldsymbol{\mu}_{Y,t}}{\sigma_Y}+\mathbf{R}_t\sigma_Y, \tag{A25}$$

where

$$\boldsymbol{\mu}_{Y,t}=\sum_{k=1}^{2}w_{k,t}\mu_{Y,k}$$

and

$$w_{k,t}=\frac{\nu_{k,t}}{\gamma_k}\mathbf{R}_t,\, k\in\{1,2\}.$$

Also,

$$r_t+\frac{1}{2}\theta_t^2=\hat{r}_{1,t}+\gamma_1\left(\mu_{\nu_1,t}-\frac{1}{2}\sigma_{\nu_1,t}^2\right)+\frac{1}{2}\hat{\theta}_{1,t}^2.$$

Therefore,

$$r_t=\hat{r}_{1,t}+\gamma_1\left(\mu_{\nu_1,t}-\frac{1}{2}\sigma_{\nu_1,t}^2\right)+\frac{1}{2}\hat{\theta}_{1,t}^2-\frac{1}{2}(\hat{\theta}_{1,t}^2+2\gamma_1\hat{\theta}_{1,t}^2\sigma_{\nu_1,t}+\gamma_1^2\sigma_{\nu_1,t}^2)$$

$$r_t=r_{1,t}+\gamma_1\left(\mu_{\nu_1,t}-\frac{1}{2}\sigma_{\nu_1,t}\right)-\gamma_1\hat{\theta}_1\sigma_{\nu_1,t}-\frac{1}{2}\gamma_1^2\sigma_{\nu_1,t}^2.$$

Substituting (A24) and $\mu_{\nu_1,t}$, $\sigma_{\nu_1,t}$ from Proposition 4 into the above expression and simplifying gives the expression in (49).

8. Proof of Corollary 3: Risk-free rate with correct beliefs or with identical risk aversions and habits

Equation (51) follows from (49) after setting $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$, and simplifying. Equation (52) follows from (49) after setting $\gamma_1 = \gamma_2 = \gamma$ and $\psi_1 = \psi_2 = \psi$, and simplifying.

9. Proof of Proposition 6: Market price of risk

We have (55) from (A25). Given that the $\xi_{k,t}$ is the exponential martingale defining the change of probability measure from $\mathbb{P}$ to $\mathbb{P}^k$, it follows from Girsanov's theorem that Agent k's perception of the market price of risk is

$$\theta_{k,t} = \theta_t + \sigma_{\xi,k}.$$

Hence,

$$\theta_{k,t} = \theta_t + \sigma_{\xi,k} = \mathbf{R}_t \sigma_Y + \frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y} + \frac{\mu_{Y,k} - \mu_Y}{\sigma_Y} = \mathbf{R}_t \sigma_Y + \frac{\mu_{Y,k} - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}.$$

Thus, we obtain (56).

10. Proof of Corollary 4: Market price of risk with correct beliefs or with identical risk aversions

Equation (57) follows from (56) after setting $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$, and simplifying. Equation (58) follows from (56) after setting $\gamma_1 = \gamma_2 = \gamma$, and simplifying.

11. Proof of Proposition 7

Equation (13) can be rewritten as

$$\nu_{2,t}^{\gamma_2} e^{q_t + d_\omega \omega_t} = \nu_{1,t}^{\gamma_1},$$

and so

$$\nu_{2,t}^{\eta} e^{\frac{q_t + d_\omega \omega_t}{\gamma_1}} = \nu_{1,t}. \tag{A26}$$

Now, recall the standard results that

$$\lim_{t\to\infty} e^{at+bZ_t} = \begin{cases} \infty, \mathbb{P}-a.s., & a>0, \\ 0, \mathbb{P}-a.s., & a<0, \end{cases}$$

and

$$\limsup_{t\to\infty} e^{bZ_t} = \infty,$$

$$\liminf_{t\to\infty} e^{bZ_t} = 0.$$

From the above results it follows that to ensure that $\lim_{t\to\infty} e^{at+bZ_t}$ is strictly between zero and infinity, we need to have both a and b equal to zero. It then follows from the expression in (A26) that both agents will survive $\mathbb{P}$-almost-surely; that is, the economy will be stationary almost surely under $\mathbb{P}$, if and only if $\mu_q = 0$ and $\sigma_q = 0$. We can also see that $\mu_q > 0$ then Agent 2 survives, but Agent 1 does not. We can therefore define survival indices as in (59).

12. Proof of Proposition 8

To prove Proposition 8, we first prove the following three lemmas.

Lemma A1 (Partial differential equation for price of a dividend strip). Given the aggregate endowment process specified in Equation (36) and the beliefs process in (37), the time-t price of the dividend strip, which pays the cash flow Y_u at date $u>t$, is given by $V^Y_{t,u}=v^Y_{t,u}Y_t$, where

$$
\begin{aligned}
0=&\frac{\partial v^Y_{t,u}}{\partial t}+\frac{\partial v^Y_{t,u}}{\partial \omega_t}\lambda_x(\bar{\omega}-\omega_t)+\nu_{1,t}\mu_{\nu_1,t}\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}\\
&+\frac{1}{2}\sigma_Y^2\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t^2}+\sigma_Y\nu_{1,t}\sigma_{\nu_1,t}\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t\partial \nu_{1,t}}+\frac{1}{2}\nu_{1,t}^2\sigma_{\nu_1,t}^2\frac{\partial^2 v^Y_{t,u}}{\partial \nu_{1,t}^2}\\
&+(\mu_Y-\theta_t\sigma_Y)v^Y_{t,u}+(\sigma_Y-\theta_t)\left(\sigma_Y\frac{\partial v^Y_{t,u}}{\partial \omega_t}+\nu_{1,t}\sigma_{\nu_1,t}\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}\right)-r_t v^Y_{t,u},
\end{aligned}
\tag{A27}
$$

and

$$v^Y_{t,u}|_{u=t}=1,$$

$$\lim_{\omega_t\to\infty}v^Y_{t,u}=\infty,\ \lim_{\omega_t\to-\infty}v^Y_{t,u}=0,\ \text{if}\ \gamma_k>\frac{1}{\psi_k},\ k\in\{1,2\},$$

$$v^Y_{t,u}|_{\nu_{k,t}=1}=e^{-\left(\hat{r}_k+\frac{1}{\psi_k}\sigma_Y^2-\mu_{Y,k}\right)(u-t)}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)\left[(1-e^{-\lambda_x(u-t)})(\hat{\omega}^Y_k-\omega_t)-\frac{1}{2}\left(\gamma_k-\frac{1}{\psi_k}\right)\frac{1-e^{-2\lambda_x(u-t)}}{2\lambda_x}\sigma_Y^2\right]},$$

where

$$\hat{\omega}^Y_k=\frac{\mu_{Y,k}+\left(1-\frac{1}{\psi_k}\right)\sigma_Y^2-\frac{1}{2}\sigma_Y^2}{\lambda_x},$$

$$\hat{r}_k=\beta_k+\frac{1}{\psi_k}\mu_{Y,k}-\frac{1}{2}\frac{1}{\psi_k}\left(1+\frac{1}{\psi_k}\right)\sigma_Y^2.$$

Proof of Lemma A1. The fundamental asset pricing equation (see Cochrane 2001) states that

$$E_t[dV^Y_{t,u}-rV^Y_{t,u}dt]=-E_t\left[\frac{d\pi_t}{\pi_t}dV^Y_{t,u}\right]. \tag{A28}$$

Because $V^Y_{t,u}=Y_t v^Y_{t,u}$, where $v^Y_{t,u}$ is a function of time and the state variables ω_t and $\nu_{1,t}$, that is, $v^Y_{t,u}=v^Y_{t,u}(t,\omega_t,\nu_{1,t})$, it follows from Ito's lemma that

$$
\begin{aligned}
\frac{dV^Y_{t,u}}{V^Y_{t,u}}=&\frac{dv^Y_{t,u}}{v^Y_{t,u}}+\frac{dY_t}{Y_t}+\frac{dv^Y_{t,u}}{v^Y_{t,u}}\frac{dY_t}{Y_t}\\
=&\frac{1}{v^Y_{t,u}}\left[\frac{\partial v^Y_{t,u}}{\partial t}+\frac{\partial v^Y_{t,u}}{\partial \omega_t}d\omega_t+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}d\nu_{1,t}+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t^2}(d\omega_t)^2+\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t\partial \nu_{1,t}}d\omega_t d\nu_{1,t}+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t^2}(d\nu_{1,t})^2\right]\\
&+\mu_Y dt+\sigma_Y dZ_t+\frac{1}{v^Y_{t,u}}\left(\frac{\partial v^Y_{t,u}}{\partial \omega_t}d\omega_t+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}d\nu_{1,t}\right)\frac{dY_t}{Y_t}.
\end{aligned}
$$

Then

$$E_t\left[\frac{dV^Y_{t,u}}{V^Y_{t,u}}\right]=\frac{1}{v^Y_{t,u}}$$

$$\times\left(\frac{\partial v^Y_{t,u}}{\partial t}+\frac{\partial v^Y_{t,u}}{\partial \omega_t}E_t[d\omega_t]+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}E_t[d\nu_{1,t}]+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t^2}(d\omega_t)^2+\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t\partial \nu_{1,t}}d\omega_t d\nu_{1,t}+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \nu_{1,t}^2}(d\nu_{1,t})^2\right)$$

$$+\mu_Y dt+\frac{1}{v^Y_{t,u}}\left(\frac{\partial v^Y_{t,u}}{\partial \omega_t}d\omega_t+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}d\nu_{1,t}\right)\frac{dY_t}{Y_t}$$

$$=\frac{1}{v^Y_{t,u}}\left(\frac{\partial v^Y_{t,u}}{\partial t}+\frac{\partial v^Y_{t,u}}{\partial \omega_t}\lambda_x(\bar{\omega}-\omega_t)+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}\nu_{1,t}\mu_{\nu_1,t}+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t^2}\sigma_Y^2+\frac{\partial^2 v^Y_{t,u}}{\partial \omega_t\partial \nu_{1,t}}\sigma_Y\nu_{1,t}\sigma_{\nu_1,t}+\frac{1}{2}\frac{\partial^2 v^Y_{t,u}}{\partial \nu_{1,t}^2}\nu_{1,t}^2\sigma^2_{\nu_1,t}\right)$$

$$+\mu_Y dt+\frac{1}{v^Y_{t,u}}\left(\frac{\partial v^Y_{t,u}}{\partial \omega_t}\sigma_Y+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_1,t}\right)\sigma_Y,$$

and

$$-E_t\left[\frac{dV^Y_{t,u}}{V^Y_{t,u}}\frac{d\pi_t}{\pi_t}\right]=\left[\frac{1}{v^Y_{t,u}}\left(\frac{\partial v^Y_{t,u}}{\partial \omega_t}\sigma_Y+\frac{\partial v^Y_{t,u}}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_1,t}\right)+\sigma_Y\right]\theta_t dt.$$

Equation (A27) then follows from (A28).

When $\omega_t\to\infty$, current marginal utility tends to zero for Agent k, $k\in\{1,2\}$ (provided $\gamma_k>\frac{1}{\psi_k}$). However, ω is mean reverting, and so future marginal utility will be higher. Consequently, the marginal rate of substitution is infinite, and so the price-dividend ratio will be infinite. When $\omega_t\to-\infty$, current marginal utility tends to ∞ for Agent k, $k\in\{1,2\}$, (provided $\gamma_k>\frac{1}{\psi_k}$). However, ω is mean reverting, and so future marginal utility will be lower. Consequently, the marginal rate of substitution is zero, and so the price-dividend ratio will be zero. Hence, $\lim_{\omega_t\to\infty}v^Y_{t,u}=\infty$ and $\lim_{\omega_t\to-\infty}v^Y_{t,u}=0$.

When $\nu_{k,t}=1$, the economy is populated solely by agents of Type k. Hence,

$$v^X_{t,u}=E_t\left[\frac{\hat{\pi}_{k,u}}{\hat{\pi}_{k,t}}\frac{X_u}{X_t}\right]$$
$$=E_t\left[\frac{\xi_{k,u}}{\xi_{k,t}}e^{-\beta_k(u-t)}e^{\left(\gamma_k-\frac{1}{\psi_k}\right)(\omega_u-\omega_t)-\frac{1}{\psi_k}(y_u-y_t)}\frac{X_u}{X_t}\right]$$
$$=e^{-\beta_k(u-t)}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)\omega_t+\frac{1}{\psi_k}y_t-x_t}E_t\left[\frac{\xi_{k,u}}{\xi_{k,t}}e^{\left(\gamma_k-\frac{1}{\psi_k}\right)\omega_u-\frac{1}{\psi_k}y_u+x_u}\right],$$

where $x=\ln X$.

We can show, after some algebra, that

$$\frac{\hat{\pi}_{k,u}}{\hat{\pi}_{k,u}}\frac{X_u}{X_t}=e^{-k_k(u-t)}\frac{M_{k,u}}{M_{k,t}}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)(\omega_u-\omega_t)},$$

where

$$k_k=\beta_k+\frac{1}{\psi_k}\mu_{Y,k}-\frac{1}{2}\frac{1}{\psi_k}\left(1+\frac{1}{\psi_k}\right)\sigma_Y^2+\frac{1}{\psi_k}\sigma_X^{sys}\sigma_Y-\mu_{X,k},$$

and $M_{k,t}$ is the following exponential martingale under $\mathbb{P}$:

$$\frac{dM_{k,t}}{M_{k,t}}=\sigma_X^{id}dZ_t^{id}+\left(\sigma_X^{sys}+\sigma_{\xi,k}-\frac{1}{\psi_k}\sigma_Y\right)dZ_t,\ M_{k,t}=1. \tag{A29}$$

We define the new probability measures $\hat{\mathbb{P}}^k$ on $(\Omega, \mathcal{F})$ via

$$\hat{\mathbb{P}}^k(A) = E(1_A M_{k,T}),\ A \in \mathcal{F}_T,\ k \in \{1,2\}. \tag{A30}$$

Hence,

$$v_{t,u}^X = E_t\left[e^{-k_k(u-t)}\frac{M_{k,u}}{M_{k,t}}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)(\omega_u-\omega_t)}\right] = e^{-k_k(u-t)}E_t^k\left[e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)(\omega_u-\omega_t)}\right].$$

Define the moment generating function

$$\mathcal{M}^k(\omega_t, u-t, a) = E_t^k\left[e^{a\omega_u}\right].$$

Because ω is Gaussian, we have

$$\begin{aligned}\mathcal{M}^k(\omega_t,u-t,a) &= e^{aE_t^k[\omega_u]+\frac{1}{2}a^2\mathrm{Var}_t^k[\omega_u]}\\ &= e^{a[e^{-\lambda_x(u-t)}\omega_t+(1-e^{-\lambda_x(u-t)})\hat{\omega}_k^X]+\frac{1}{2}a^2\frac{1-2\lambda_x(u-t)}{2\lambda_x}\sigma_Y^2},\end{aligned}$$

where

$$\hat{\omega}_k^X = \frac{\mu_{Y,k}+\left(\sigma_X^{sys}-\frac{1}{\psi_k}\sigma_Y\right)\sigma_Y-\frac{1}{2}\sigma_Y^2}{\lambda_x}.$$

Hence,

$$\begin{aligned}v_{t,u}^X &= e^{-k_k(u-t)}e^{\left(\gamma_k-\frac{1}{\psi_k}\right)\omega_t}\mathcal{M}^k\left(\omega_t,u-t,-\left(\gamma_k-\frac{1}{\psi_k}\right)\right)\\ &= e^{-k_k(u-t)}e^{\left(\gamma_k-\frac{1}{\psi_k}\right)\omega_t}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)\left\{[e^{-\lambda_x(u-t)}\omega_t+(1-e^{-\lambda_x(u-t)})\hat{\omega}_k^X]-\frac{1}{2}\left(\gamma_k-\frac{1}{\psi_k}\right)\frac{1-e^{-2\lambda_x(u-t)}}{2\lambda_x}\sigma_Y^2\right\}}\\ &= e^{-k_k(u-t)}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)\left[(1-e^{-\lambda_x(u-t)})(\hat{\omega}_k^X-\omega_t)-\frac{1}{2}\left(\gamma_k-\frac{1}{\psi_k}\right)\frac{1-e^{-2\lambda_x(u-t)}}{2\lambda_x}\sigma_Y^2\right]}.\end{aligned}$$

Therefore,

$$v_{t,u}^Y = e^{-k_k(u-t)}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)\left[(1-e^{-\lambda_x(u-t)})(\hat{\omega}_k^Y-\omega_t)-\frac{1}{2}\left(\gamma_k-\frac{1}{\psi_k}\right)\frac{1-e^{-2\lambda_x(u-t)}}{2\lambda_x}\sigma_Y^2\right]},$$

where now

$$k_k = \beta_k + \frac{1}{\psi_k}\mu_{Y,k} - \frac{1}{2}\frac{1}{\psi_k}\left(1+\frac{1}{\psi_k}\right)\sigma_Y^2 + \frac{1}{\psi_k}\sigma_Y^2 - \mu_{Y,k},$$

and

$$\hat{\omega}_k^Y = \frac{\mu_{Y,k}+\left(1-\frac{1}{\psi_k}\right)\sigma_Y^2-\frac{1}{2}\sigma_Y^2}{\lambda_x}.$$

Although we could solve (A27) numerically, an analytical approach would seem difficult, given the relatively complicated nonlinear dependence of $\mu_{\nu_1,t}$ and $\sigma_{\nu_1,t}$ on $\nu_{1,t}$. Nevertheless, we can find an analytical solution by using Proposition 3. This relies on finding only the value of the claims $\phi_{n,1,t,u}^Y$ and $\phi_{n,2,t,u}^Y$ defined in Equations (30) and (31), which pay Y_u at u whenever $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R$ and $\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R$, respectively. The proof uses the following lemmas.

Lemma A2 (Values of $\phi^X_{n,1,t}$ and $\phi^X_{n,2,t}$).

$$\phi^X_{n,1,t} = e^{-k_1(T-t)} e^{-a_1 q_t - b_1 \omega_t}\, \hat{E}^1_t\left[e^{a_1 q_T + b_1 \omega_T}\, 1_{\{q_T + d_\omega \omega_T > \rho\}}\right]$$

$$\phi^X_{n,2,t} = e^{-k_2(T-t)} e^{-a_2 q_t - b_2 \omega_t}\, \hat{E}^2_t\left[e^{a_2 q_T + b_2 \omega_T}\, 1_{\{q_T + d_\omega \omega_T < \rho\}}\right],$$

where

$$\hat{r}_1 = \beta_1 + \frac{1}{\psi_1}\mu_{Y,1} - \frac{1}{2}\frac{1}{\psi_1}\left(1 + \frac{1}{\psi_1}\right)\sigma_Y^2,$$

$$\hat{r}_2 = \beta_2 + \frac{1}{\psi_2}\mu_{Y,2} - \frac{1}{2}\frac{1}{\psi_2}\left(1 + \frac{1}{\psi_2}\right)\sigma_Y^2,$$

$$\hat{k}_1 = \hat{r}_1 + \frac{1}{\psi_1}\sigma_X^{sys}\sigma_Y - \mu_{X,1}, \tag{A31}$$

$$\hat{k}_2 = \hat{r}_2 + \frac{1}{\psi_2}\sigma_X^{sys}\sigma_Y - \mu_{X,2}, \tag{A32}$$

$$a_1 = -\frac{n}{\gamma_2}, \tag{A33}$$

$$b_1 = -\left[\frac{n}{\gamma_2}\left(\gamma_2 - \frac{1}{\psi_2}\right) + \left(1 - \frac{n}{\gamma_2}\right)\left(\gamma_1 - \frac{1}{\psi_1}\right)\right],$$

$$a_2 = \frac{n}{\gamma_1},$$

$$b_2 = -\left[\left(1 - \frac{n}{\gamma_2}\right)\left(\gamma_2 - \frac{1}{\psi_2}\right) + \frac{n}{\gamma_1}\left(\gamma_1 - \frac{1}{\psi_1}\right)\right], \tag{A34}$$

$$\rho = \ln R, \tag{A35}$$

and $\hat{E}^k_t[\cdot]$ is the date-t conditional expectation operator with respect to the probability measure $\hat{\mathbb{P}}^k$ defined in (A30).

Proof of Lemma A2. Recall that

$$\phi^X_{n,1,t,T} = E_t\left[\frac{\hat{\pi}_{1,T}^{1-\frac{n}{\gamma_2}}\,\hat{\pi}_{2,T}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\,\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}\,\frac{X_T}{X_t}\,1_{\left\{\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}} > R\right\}}\right], n \in \mathbb{N}_0, \tag{A36}$$

$$\phi^X_{n,2,t,T} = E_t\left[\frac{\hat{\pi}_{1,T}^{\frac{n}{\gamma_1}}\,\hat{\pi}_{2,T}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\,\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\,\frac{X_T}{X_t}\,1_{\left\{\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}} < R\right\}}\right], n \in \mathbb{N}_0. \tag{A37}$$

Simplifying (A36) gives

$$\phi^X_{n,1,t,T} = E_t\left[\frac{\hat{\pi}_{1,T}^{1-\frac{n}{\gamma_2}}\,\hat{\pi}_{2,T}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\,\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}\,\frac{X_T}{X_t}\,1_{\left\{\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}} > R\right\}}\right]$$

$$= E_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{-\frac{n}{\gamma_2}}\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{1,t}}\,\frac{X_T}{X_t}\,1_{\left\{\ln\left(\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}}\right) > \rho\right\}}\right],$$

where ρ is defined in (A35). Simplifying (A37) gives

$$\phi^X_{n,2,t,T} = E_t\left[\frac{\hat{\pi}_{1,T}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,T}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\frac{X_T}{X_t}1_{\left\{\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}}<R\right\}}\right]$$

$$= E_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}}\frac{\hat{\pi}_{2,T}}{\hat{\pi}_{2,t}}\frac{X_T}{X_t}1_{\left\{\ln\left(\frac{\hat{\pi}_{1,T}}{\hat{\pi}_{2,T}}\right)<\rho\right\}}\right].$$

Note that

$$\frac{\hat{\pi}_{k,u}}{\hat{\pi}_{k,u}}\frac{X_u}{X_t} = e^{-k_k(u-t)}\frac{M_{k,u}}{M_{k,t}}e^{-\left(\gamma_k-\frac{1}{\psi_k}\right)(\omega_T-\omega_t)},$$

where $k_k, k\in\{1,2\}$ are defined in (A31) and (A32), and $M_{k,t}, k\in\{1,2\}$, is defined in (A29). Hence,

$$\phi^X_{n,1,t,T} = E_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{-\frac{n}{\gamma_2}}e^{-k_1(T-t)}\frac{M_{1,T}}{M_{1,t}}e^{-\left(\gamma_1-\frac{1}{\psi_1}\right)(\omega_T-\omega_t)}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]$$

$$\phi^X_{n,2,t,T} = E_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}}e^{-k_2(T-t)}\frac{M_{2,T}}{M_{2,t}}e^{-\left(\gamma_2-\frac{1}{\psi_2}\right)(\omega_T-\omega_t)}1_{\{q_T+d_\omega\omega_T<\rho\}}\right].$$

Therefore,

$$\phi^X_{n,1,t} = \hat{E}^1_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{-\frac{n}{\gamma_2}}e^{-k_1(T-t)}e^{-\left(\gamma_1-\frac{1}{\psi_1}\right)(\omega_T-\omega_t)}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]$$

$$= e^{-k_1(T-t)}e^{-a_1q_t-b_1\omega_t}\hat{E}^1_t\left[e^{a_1q_T+b_1\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right],$$

and

$$\phi^X_{n,2,t} = \hat{E}^2_t\left[\left(\frac{\hat{\pi}_{1,T}/\hat{\pi}_{2,T}}{\hat{\pi}_{1,t}/\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}}e^{-k_2(T-t)}e^{-\left(\gamma_2-\frac{1}{\psi_2}\right)(\omega_T-\omega_t)}1_{\{q_T+d_\omega\omega_T<\rho\}}\right]$$

$$= e^{-k_2(T-t)}e^{-a_2q_t-b_2\omega_t}\hat{E}^2_t\left[e^{a_2q_T+b_2\omega_T}1_{\{q_T+d_\omega\omega_T<\rho\}}\right],$$

where a_1, b_1, a_2, and b_2 are defined in (A33)–(A34).

We shall need some basic definitions for Fourier transforms to prove Lemma A3.

Definition A4. For an integrable function $f:\mathbb{R}\to\mathbb{C}$, its Fourier transform is given by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ikx}f(k)dk, \forall x\in\mathbb{R}. \tag{A38}$$

We shall use the notation

$$\mathcal{F}[f(k),x] = \int_{-\infty}^{\infty} e^{ikx}f(k)dk. \tag{A39}$$

Under certain conditions (see, for example, Friedlander and Joshi 1999) $f(k)$ can be reconstructed from $\mathcal{F}[f(k),x]$ via the *inverse* Fourier transform

$$f(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{f}(x)dx,$$

for which we use the notation

$$\mathcal{F}^{-1}[\hat{f}(x),k] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{f}(x)dx.$$

The following Lemma provides a closed-form expression for the moment generating function of the two-dimensional affine process, (q_t,ω_t), under the probability measure $\hat{\mathbb{P}}^k$.

Lemma A3 (Moment generating function).

$$\hat{E}_t^k\left[e^{a_k q_T + b_k \omega_T} 1_{\{q_T + d_\omega \omega_T > \rho\}}\right] = \mathcal{M}_k^X(q_t, \omega_t, T-t, a_k, b_k)\Phi\left(\frac{B_k^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_k(q_t,\omega_t,T-t)}}\right), \quad \text{(A40)}$$

where the moment generating function, $\mathcal{M}_k^X(q_t, \omega_t, T-t, a, b)$, defined by

$$\mathcal{M}_k^X(q_t,\omega_t,T-t,a,b) = \hat{E}_t^k[e^{aq_T + b\omega_T}], \quad \text{(A41)}$$

is given by

$$\mathcal{M}_k^X(q_t,\omega_t,T-t,a,b) = \exp\left\{a[q_t + \hat{\mu}_{q,k}^X(T-t)] + b[e^{-\lambda_x(T-t)}\omega_t + (1-e^{-\lambda_x(T-t)})\hat{\omega}_k^X]\right. \quad \text{(A42)}$$

$$\left. + \frac{1}{2}a^2\sigma_q^2(T-t) + ab\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y + \frac{1}{2}b^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2\right\},$$

and

$$A_k(q_t,\omega_t,T-t) = \frac{1}{2}\left[\sigma_q^2(T-t) + 2d_\omega\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y + d_\omega^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2\right], \quad \text{(A43)}$$

$$B_k^X(q_t,\omega_t,T-t) = q_t + \hat{\mu}_{q,k}^X(T-t) + d_\omega[e^{-\lambda_x(T-t)}\omega_t + (1-e^{-\lambda_x(T-t)})\hat{\omega}_k^X] \quad \text{(A44)}$$

$$+ a_k\sigma_q^2(T-t) + (b_k + a_k d_\omega)\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y + b_k d_\omega\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2,$$

where

$$\mu_q = (\beta_2 - \beta_1) + \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right),$$

$$\sigma_q = \sigma_{\xi,1} - \sigma_{\xi,2} + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\sigma_Y,$$

$$\hat{\mu}_{q,k}^X = \mu_q + \left(\sigma_X^{sys} + \sigma_{\xi,k} - \frac{1}{\psi_k}\sigma_Y\right)\sigma_q,$$

$$\hat{\omega}_k^X = \frac{\mu_{Y,k} - \frac{1}{2}\sigma_Y^2 + \left(\sigma_X^{sys} - \frac{1}{\psi_k}\sigma_Y\right)\sigma_Y}{\lambda_x}, \quad \text{(A45)}$$

$$d_\omega = \left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right).$$

Proof of Lemma A3. First, we prove (A42). Because (q_T, ω_T) is Gaussian, we have

$$\mathcal{M}_k^X(q_t,\omega_t,T-t,a,b) = e^{\hat{E}^k[aq_T + b\omega_T] + \frac{1}{2}\widehat{\mathrm{Var}}_t^k[aq_T + b\omega_T]},$$

where $\widehat{\mathrm{Var}}_t^k$ is the time-t conditional variance operator under the probability measure $\widehat{\mathbb{P}}^k$. Note that

$$\hat{E}^k[aq_T + b\omega_T] = a\hat{E}^k[q_T] + b\hat{E}^k[\omega_T]$$

$$= a[q_t + \hat{\mu}_{q,k}^X(T-t)] + b[e^{-\lambda_x(T-t)}\omega_t + (1-e^{-\lambda_x(T-t)})\hat{\omega}_k^X]$$

and

$$\frac{1}{2}\widehat{\mathrm{Var}}_t^k[aq_T+b\omega_T]=\frac{1}{2}a^2\widehat{\mathrm{Var}}_t^k[q_T]+ab\widehat{\mathrm{Cov}}_t^k[q_T,\omega_T]+\frac{1}{2}b^2\widehat{\mathrm{Var}}_t^k[\omega_T]$$
$$=\frac{1}{2}a^2\sigma_q^2(T-t)+ab\,\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y+\frac{1}{2}b^2\,\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2,$$

where the long-run mean of ω under the probability measure $\hat{\mathbb{P}}^k$ is given by $\hat{\omega}_k^X$, defined in (A45). Hence,

$$\mathcal{M}_k^X(q_t,\omega_t,T-t,a,b)=\exp\left\{a[q_t+\hat{\mu}_{q,k}^X(T-t)]+b[e^{-\lambda_x(T-t)}\omega_t+(1-e^{-\lambda_x(T-t)})\hat{\omega}_k^X]\right.$$
$$\left.+\frac{1}{2}a^2\sigma_q^2(T-t)+ab\,\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_Y+\frac{1}{2}b^2\,\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_Y^2\right\}.$$

Our proof of (A40) relies on using Fourier transforms (see Heston 1993; Duffie, Pan, and Singleton 2000). Taking the Fourier transform of $\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]$ gives (see (A38) and (A39) for relevant definitions)

$$\mathcal{F}\left[\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right],x\right]=\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}\mathcal{F}[1_{\{q_T+d_\omega\omega_T>\rho\}},x]\right]$$
$$=\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}\mathcal{F}[\theta(q_T+d_\omega\omega_T-\rho),x]\right],$$

where $\theta(z)$ is the Heaviside step function, defined by

$$\theta(z)=\begin{cases}0, & z<0\\ \frac{1}{2}, & z=0\\ 1, & z>0.\end{cases}$$

Using the standard result that

$$\mathcal{F}[\theta(q_T+d_\omega\omega_T-\rho),x]=\frac{e^{ix(q_T+d_\omega\omega_T)}}{ix}+\pi\delta(x),$$

where $\delta(z)$ is the Dirac-delta function. It follows that

$$\mathcal{F}\left[\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right],x\right]=\hat{E}_t^k\left[e^{a_kq_T+b_k\omega_T}\left(\frac{e^{ix(q_T+d_\omega\omega_T)}}{ix}+\pi\delta(x)\right)\right]$$
$$=\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix}$$
$$+\pi\delta(x)\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k),$$

where $\mathcal{M}_k^X(q_t,\omega_t,T-t,a,b)$ is defined in (A41). Taking the inverse Fourier transform, we obtain

$$\hat{E}_t^k\left[e^{a_k q_T+b_k\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]=\mathcal{F}^{-1}\left[\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix},\rho\right]$$
$$+\pi\mathcal{F}^{-1}[\delta(x),\rho]\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\rho x}\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix}dx$$
$$+\pi\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k)\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\rho x}\delta(x)dx$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\rho x}\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix}dx$$
$$+\frac{1}{2}\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k), \tag{A46}$$

where we have used the standard result that

$$\int_{-\infty}^{\infty}e^{-i\rho x}\delta(x)dx=1.$$

We now show that

$$\int_{-\infty}^{\infty}e^{-i\rho x}\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix}dx$$
$$=2\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k)\int_0^{\infty}e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}dx, \tag{A47}$$

where $A_k(q_t,\omega_t,T-t)$ and $B_k^X(q_t,\omega_t,T-t)$ are defined in (A43) and (A44), respectively. We start by observing that

$$\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)=\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k)e^{-A_k(q_t,\omega_t,T-t)x^2+iB_k^X(q_t,\omega_t,T-t)x}.$$

Hence,

$$\int_{-\infty}^{\infty}e^{-i\rho x}\frac{\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k+ix,b_k+ixd_\omega)}{ix}dx$$
$$=\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k)\int_{-\infty}^{\infty}e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{e^{i(B_k^X(q_t,\omega_t,T-t)-\rho)x}}{ix}dx$$
$$=\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k) \tag{A48}$$
$$\times\int_{-\infty}^{\infty}e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\cos[(B_k^X(q_t,\omega_t,T-t)-\rho)x]+i\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{ix}dx.$$

Because $e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\cos[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}$ and $e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}$ are odd and even functions of x, respectively, it follows that

$$\int_{-\infty}^{\infty}e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\cos[(B_k^X(q_t,\omega_t,T-t)-\rho)x]+i\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{ix}dx$$
$$=2\int_0^{\infty}e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}dx.$$

Hence, (A47) follows from (A48).

Together with Equation (A47), Equation (A46) implies that

$$\hat{E}_t^k\left[e^{a_k q_T+b_k\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]=\frac{1}{\pi}\int_0^\infty e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}dx$$
$$+\frac{1}{2}\mathcal{M}_k^X(q_t,\omega_t,T-t,a_k,b_k). \qquad \text{(A49)}$$

Because

$$\int_0^\infty e^{-A_k(q_t,\omega_t,T-t)x^2}\frac{\sin[(B_k^X(q_t,\omega_t,T-t)-\rho)x]}{x}dx=\frac{1}{2}\pi\left[2\Phi\left(\frac{B_k^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_k(q_t,\omega_t,T-t)}}\right)-1\right],$$

where $\Phi(\cdot)$ is the standard cumulative normal distribution function (a result which can be obtained using *Mathematica*), (A49) implies (A40).

To prove Proposition 8, our first step is to use Lemma A2; we then use Lemma A3 to simplify (A36) and (A37):

$$\hat{E}_t^1\left[e^{a_1 q_T+b_1\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]=\mathcal{M}_1^X(q_t,\omega_t,T-t,a_1,b_1)\Phi\left(\frac{B_1^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right),$$

and

$$\begin{aligned}\hat{E}_t^2\left[e^{a_2 q_T+b_2\omega_T}1_{\{q_T+d_\omega\omega_T<\rho\}}\right]&=\hat{E}_t^2\left[e^{a_2 q_T+b_2\omega_T}\left(1-1_{\{q_T+d_\omega\omega_T>\rho\}}\right)\right]\\&=\hat{E}_t^2\left[e^{a_2 q_T+b_2\omega_T}\right]-\hat{E}_t^2\left[e^{a_2 q_T+b_2\omega_T}1_{\{q_T+d_\omega\omega_T>\rho\}}\right]\\&=\mathcal{M}_2^X(q_t,\omega_t,T-t,a_2,b_2)\left[1-\Phi\left(\frac{B_2^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right)\right]\\&=\mathcal{M}_2^X(q_t,\omega_t,T-t,a_2,b_2)\Phi\left(\frac{\rho-B_2^X(q_t,\omega_t,T-t)}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right).\end{aligned}$$

Hence,

$$\phi_{n,1,t,T}^X=e^{-k_1(T-t)}e^{-a_1 q_t-b_1\omega_t}\mathcal{M}_1^X(q_t,\omega_t,T-t,a_1,b_1)\Phi\left(\frac{B_1^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right),$$

$$\phi_{n,2,t,T}^X=e^{-k_2(T-t)}e^{-a_2 q_t-b_2\omega_t}\mathcal{M}_2^X(q_t,\omega_t,T-t,a_2,b_2)\Phi\left(\frac{\rho-B_2^X(q_t,\omega_t,T-t)}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right).$$

We know that

$$q_t=\ln\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}-d_\omega\omega_t=\ln\left[\frac{(1-\nu_{1,t})^{-\gamma_2}}{\nu_{1,t}^{-\gamma_1}}\right]-d_\omega\omega_t,$$

and so

$$\phi_{n,1,t,T}^X=e^{-k_1(T-t)-(b_1-a_1 d_\omega)\omega_t}\left(\frac{\nu_{2,t}^{\gamma_2}}{\nu_{1,t}^{\gamma_1}}\right)^{a_1}\mathcal{M}_1^X(q_t,\omega_t,T-t,a_1,b_1)\Phi\left(\frac{B_1^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right)$$

$$\phi_{n,2,t,T}^X=e^{-k_2(T-t)-(b_2-a_2 d_\omega)\omega_t}\left(\frac{\nu_{2,t}^{\gamma_2}}{\nu_{1,t}^{\gamma_1}}\right)^{a_2}\mathcal{M}_2^X(q_t,\omega_t,T-t,a_2,b_2)\Phi\left(\frac{\rho-B_2^X(q_t,\omega_t,T-t)}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right).$$

Simplifying the above expressions gives

$$\phi^X_{n,1,t,T} = e^{-k_1(T-t)+b_1(\omega_t-\hat{\omega}^X_k)[1-e^{-\lambda_x(T-t)}]}\exp\left\{a_1\hat{\mu}^X_{q,1}(T-t)+\frac{1}{2}a_1^2\sigma_q^2(T-t)\right.$$

$$\left.+a_1b_1\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_X^{sys}+\frac{1}{2}b_1^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_X^2\right\}\Phi\left(\frac{B_1^X(q_t,\omega_t,T-t)-\rho}{\sqrt{2A_1(q_t,\omega_t,T-t)}}\right),$$

$$\phi^X_{n,2,t,T} = e^{-k_2(T-t)+b_2(\omega_t-\hat{\omega}^X_k)[1-e^{-\lambda_x(T-t)}]}\exp\left\{a_2\hat{\mu}^X_{q,2}(T-t)+\frac{1}{2}a_2^2\sigma_q^2(T-t)\right.$$

$$\left.+a_2b_2\frac{1-e^{-\lambda_x(T-t)}}{\lambda_x}\sigma_q\sigma_X^{sys}+\frac{1}{2}b_2^2\frac{1-e^{-2\lambda_x(T-t)}}{2\lambda_x}\sigma_X^2\right\}\Phi\left(\frac{\rho-B_2^X(q_t,\omega_t,T-t)}{\sqrt{2A_2(q_t,\omega_t,T-t)}}\right).$$

Replacing the cashflow X with the aggregate endowment Y gives (60) and (61).

13. Proof of Proposition 9: Risk premium and volatility of risky assets

We shall derive results for a more general risky asset, which is a perpetual claim to the cash flow process, X, where the evolution of X is given by

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X^{sys} dZ_t + \sigma_X^{id} dZ_t^{id}, \tag{A50}$$

where Z_t^{id} is a standard Brownian motion under $\mathbb{P}$, orthogonal to Z_t. Under probability measure $\mathbb{P}^k, k\in\{1,2\}$, the dynamics of the cash flow process are given by

$$\frac{dX_t}{X_t} = \mu_{X,k} dt + \sigma_X^{sys} dZ_{k,t} + \sigma_X^{id} dZ_t^{id},$$

where $\mu_{X,k}$ is given by

$$\frac{\mu_{X,k}-\mu_X}{\sigma_X^{sys}} = \frac{\mu_{Y,k}-\mu_Y}{\sigma_Y}.$$

Then, to get the risk premium and the volatility of the stock market, we will set $\mu_X=\mu_Y$, $\sigma_X^{sys}=\sigma_Y$, and $\sigma_X^{id}=0$.

The risk premium for the claim paying X in perpetuity is given by the standard asset pricing equation:

$$E_t\left[\frac{dP_t^X+X_t dt}{P_t^X}-r_t dt\right] = -E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t^X}{P_t^X}\right]. \tag{A51}$$

Applying Ito's lemma to $P_t^X = X_t p_t^X$ gives

$$\frac{dP_t^X}{P_t^X} = \frac{dX_t}{X_t}+\frac{dp_t^X}{p_t^X}+\frac{dX_t}{X_t}\frac{dp_t^X}{p_t^X}$$

$$=\mu_X dt+\sigma_X^{sys}dZ_t+\sigma_X^{id}dZ_t^{id}+\frac{1}{p_t^X}\left[\frac{\partial p_t^X}{\partial \nu_{1,t}}\nu_{1,t}(\mu_{\nu_{1,t}}dt+\sigma_{\nu_{1,t}}dZ_t)\right.$$

$$\left.+\frac{\partial p_t^X}{\partial \omega_t}(\lambda_x(\bar{\omega}-\omega_t)dt+\sigma_Y dZ_t)\right]+\frac{1}{2}\frac{1}{p_t^X}\frac{\partial^2 p_t^X}{\partial \nu_{1,t}^2}\nu_{1,t}^2\sigma_{\nu_{1,t}}^2 dt$$

$$+\sigma_X^{sys}\frac{1}{p_t^X}\frac{\partial p_t^X}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_{1,t}}dt++\sigma_X^{sys}\frac{1}{p_t^X}\frac{\partial p_t^X}{\partial \omega_t}\sigma_Y dt+\frac{1}{2}\frac{1}{p_t^X}\frac{\partial^2 p_t^X}{\partial \omega_t^2}\sigma_Y^2 dt. \tag{A52}$$

Thus, the total volatility of the return on the claim that pays X in perpetuity, $\sigma_{R,t}^{X}$, is given by

$$\sigma_{R,t}^{X} = \sqrt{\left(\sigma_{R,t}^{X,sys}\right)^2 + \left(\sigma_{R,t}^{X,id}\right)^2},$$

where the idiosyncratic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,id} = \sigma_{X}^{id},$$

and the systematic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,sys} = \sigma_{X}^{sys} + \sigma_{\nu_1,t} \frac{\nu_{1,t}}{p_t^X} \frac{\partial p_t^X}{\partial \nu_{1,t}} + \sigma_Y \frac{1}{p_t^X} \frac{\partial p_t^X}{\partial \omega_t}.$$

Hence, substituting (A52) into (A51) gives

$$\mu_{R,t}^{X} - r_t = \theta_t \sigma_{R,t}^{X,sys}, \tag{A53}$$

where

$$\mu_{R,t}^{X} dt = E_t \left[\frac{dP_t^X + X_t dt}{P_t^X} \right].$$

Substituting (55) into (A53) gives

$$\mu_{R,t}^{X} - r_t = \left(\mathbf{R}_t \sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y} \right] \right) \sigma_{R,t}^{X,sys}.$$

Also, Agent k's perception of the risk premium for the claim paying X in perpetuity is given by the standard asset pricing equation:

$$E_t^k \left[\frac{dP_t^X + X_t dt}{P_t^X} - r_t dt \right] = -E_t^k \left[\frac{d\pi_{k,t}}{\pi_{k,t}} \frac{dP_t^X}{P_t^X} \right].$$

Hence,

$$\mu_{R,k,t}^{X} - r_t = \theta_{k,t} \sigma_{R,t}^{X,sys}, \tag{A54}$$

where

$$\mu_{R,k,t}^{X} dt = E_t^k \left[\frac{dP_t^X + X_t dt}{P_t^X} \right].$$

Substituting (56) into (A54) gives

$$\mu_{R,1,t}^{X} - r_t = \left(\mathbf{R}_t \sigma_Y + \left[\frac{\mu_Y - \mu_{Y,1}}{\sigma_Y} \right] \right) \sigma_{R,t}^{X,sys},$$

Agent 2's perception of the risk premium is given by

$$\mu_{R,2,t}^{X} - r_t = \left(\mathbf{R}_t \sigma_Y + \left[\frac{\mu_Y - \mu_{Y,2}}{\sigma_Y} \right] \right) \sigma_{R,t}^{X,sys},$$

Setting $\mu_X = \mu_Y$, $\sigma_X^{sys} = \sigma_Y$, and $\sigma_X^{id} = 0$ in the above expressions gives the results in the proposition.

14. Proof of Proposition 10

$$\phi_{n,1,t,u} = E_t\left[\left(\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{1,t}}\right)^{1-\frac{n}{\gamma_1}}\left(\frac{\hat{\pi}_{2,u}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_2}}\frac{X_u}{X_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}\right]$$

$$= E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)(\varsigma_{1,u}-\varsigma_{1,t})+\frac{n}{\gamma_2}(\varsigma_{2,u}-\varsigma_{2,t})+x_u-x_t}1_{\left\{\varsigma_{1,u}-\varsigma_{2,u}>\rho\right\}}\right].$$

Taking the Fourier transform of the above expression yields

$$\mathcal{F}[\phi_{n,1,t,u},z] = E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)(\varsigma_{1,u}-\varsigma_{1,t})+\frac{n}{\gamma_2}(\varsigma_{2,u}-\varsigma_{2,t})+x_u-x_t}\mathcal{F}\left[1_{\left\{\varsigma_{1,u}-\varsigma_{2,u}>\rho\right\}},x\right]\right]$$

$$= E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)(\varsigma_{1,u}-\varsigma_{1,t})+\frac{n}{\gamma_2}(\varsigma_{2,u}-\varsigma_{2,t})+x_u-x_t}\left(\frac{e^{iz(\varsigma_{1,u}-\varsigma_{2,u})}}{iz}+\pi\delta(z)\right)\right]$$

$$= \frac{1}{iz}E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)(\varsigma_{1,u}-\varsigma_{1,t})+\frac{n}{\gamma_2}(\varsigma_{2,u}-\varsigma_{2,t})+iz(\varsigma_{1,u}-\varsigma_{2,u})+x_u-x_t}\right]$$

$$+\pi\delta(z)E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)(\varsigma_{1,u}-\varsigma_{1,t})+\frac{n}{\gamma_2}(\varsigma_{2,u}-\varsigma_{2,t})+x_u-x_t}\right]$$

$$= e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t}\left(\frac{1}{iz}E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,u}+\frac{n}{\gamma_2}\varsigma_{2,u}+iz(\varsigma_{1,u}-\varsigma_{2,u})+x_u}\right]\right.$$

$$\left.+\pi\delta(z)E_t\left[e^{\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,u}+\frac{n}{\gamma_2}\varsigma_{2,u}+x_u}\right]\right)$$

$$= e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t}\left(\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)\right.$$

$$\left.+\pi\delta(z)\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,1-\frac{n}{\gamma_1},\frac{n}{\gamma_2},1\right)\right).$$

Taking the inverse Fourier transform gives

$$\phi_{n,1,t,u} = e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\right.$$

$$\left.+\pi\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\rho z}\delta(z)dz\,\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,1-\frac{n}{\gamma_1},\frac{n}{\gamma_2},1\right)\right]$$

$$= e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\right.$$

$$\left.+\frac{1}{2}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,1-\frac{n}{\gamma_1},\frac{n}{\gamma_2},1\right)\right],$$

and so we obtain

$$\phi_{n,1,t,u} = e^{-\left[\left(1-\frac{n}{\gamma_1}\right)\varsigma_{1,t}+\frac{n}{\gamma_2}\varsigma_{2,t}\right]-x_t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\right.$$

$$\left.+\frac{1}{2}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,1-\frac{n}{\gamma_1},\frac{n}{\gamma_2},1\right)\right]. \tag{A55}$$

Now observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\\
&=\int_{-\infty}^{0} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\\
&\quad+\int_{0}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\\
&=\int_{0}^{\infty} e^{-i\rho z}\frac{1}{-iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)-iz,\frac{n}{\gamma_2}+iz,1\right)dz\\
&\quad+\int_{0}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz.
\end{aligned}$$

If $(\varsigma_{1,t},\varsigma_{2,t},x_t)$ is analytic affine, then $\mathcal{M}(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,z_1,z_2,z_3)$, $z_1,z_2,z_3\in\mathbb{C}$ is a holomorphic function whose restriction to the real numbers is real valued, and so

$$\overline{\mathcal{M}}(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,z_1,z_2,z_3)=\mathcal{M}(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\bar{z}_1,\bar{z}_2,\bar{z}_3),$$

where $\bar{z}$ denotes the complex conjugate. Therefore,

$$\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)-iz,\frac{n}{\gamma_2}+iz,1\right)=\overline{\mathcal{M}}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right),$$

and so

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\\
&=\int_{0}^{\infty} \overline{e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)}dz\\
&\quad+\int_{0}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)dz\\
&=2\int_{0}^{\infty} Re\left[e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)\right]dz\\
&=2\int_{0}^{\infty} \frac{1}{z}Im\left[e^{i\rho z}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\left(1-\frac{n}{\gamma_1}\right)+iz,\frac{n}{\gamma_2}-iz,1\right)\right]dz.
\end{aligned}$$

Therefore, we obtain (68). Recall that

$$\phi_{n,2,t,u}=E_t\left[\left(\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_1}}\left(\frac{\hat{\pi}_{2,u}}{\hat{\pi}_{2,t}}\right)^{1-\frac{n}{\gamma_2}}\frac{X_u}{X_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}\right].$$

Hence

$$\phi_{n,2,t,u}=E_t\left[\left(\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_1}}\left(\frac{\hat{\pi}_{2,u}}{\hat{\pi}_{2,t}}\right)^{1-\frac{n}{\gamma_2}}\frac{X_u}{X_t}\right]-E_t\left[\left(\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_1}}\left(\frac{\hat{\pi}_{2,u}}{\hat{\pi}_{2,t}}\right)^{1-\frac{n}{\gamma_2}}\frac{X_u}{X_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}\right].$$

Following the same steps as the derivation of (A55), we obtain

$$\phi_{n,2,t,u} = e^{-\left[\frac{n}{\gamma_1}\varsigma_{1,t}+\left(1-\frac{n}{\gamma_2}\right)\varsigma_{2,t}\right]-x_t}\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\rho z}\frac{1}{iz}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\frac{n}{\gamma_1}+iz,1-\frac{n}{\gamma_2}-iz,1\right)dz + \frac{1}{2}\mathcal{M}\left(\varsigma_{1,t},\varsigma_{2,t},x_t,u-t,\frac{n}{\gamma_1},1-\frac{n}{\gamma_2},1\right)\right].$$

If $(\varsigma_{1,t},\varsigma_{2,t},x_t)$ is analytic affine, then we obtain (69).

15. The distribution of the consumption share

In this section we give the conditional probability density function of the consumption share ν_1, and derive its long-run behavior.

Proposition A1 (Density function for the consumption share). The density function for $\nu_{1,t+u}$, conditional on q_t and ω_t is denoted by $p_{\nu_{1,t+u}}(v|q_t,\omega_t)$, and is given by

$$p_{\nu_{1,t+u}}(v|q_t,\omega_t) = \frac{1}{\sqrt{\sigma_q^2 u + 2d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x}\sigma_q\sigma_Y + d_\omega^2\frac{1-e^{-2\lambda_x u}}{2\lambda_x}\sigma_Y^2}} \times \phi\left(\frac{\ln\frac{h_1(v)}{h_1(\nu_{1,t})} - [\mu_q u + d_\omega(1-e^{-\lambda_x u})(\bar{\omega}-\omega_t)]}{\sqrt{\sigma_q^2 u + 2d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x}\sigma_q\sigma_Y + d_\omega^2\frac{1-e^{-2\lambda_x u}}{2\lambda_x}\sigma_Y^2}}\right)\frac{h_1'(v)}{h_1(v)}, \tag{A56}$$

where $\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is the standard normal density function and

$$\mu_q = (\beta_2-\beta_1)+\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,1}^2)+\left(\frac{1}{\psi_2}-\frac{1}{\psi_1}\right)\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right),$$

$$\sigma_q = \sigma_{\xi,1}-\sigma_{\xi,2}+\left(\frac{1}{\psi_2}-\frac{1}{\psi_1}\right)\sigma_Y$$

$$d_\omega = \left(\gamma_2-\frac{1}{\psi_2}\right)-\left(\gamma_1-\frac{1}{\psi_1}\right)$$

$$h_1(v) = v^{\gamma_1}(1-v)^{-\gamma_2},$$

$$\mathbf{R}_t(v) = \left(v\frac{1}{\gamma_1}+(1-v)\frac{1}{\gamma_2}\right)^{-1}.$$

If both agents have the same survival indices, that is, $\mu_q=0$, then

$$\lim_{u\to\infty} p_{\nu_{1,t+u}}(v|\nu_{1,t}) = \frac{1}{2}(\delta(v)+\delta(v-1)),$$

where $\delta(\cdot)$ is the Dirac-delta function.

Proof of Proposition A1.
Note that

$$e^{\Delta_t} = h_1(\nu_{1,t}).$$

The cumulative distribution function for $\nu_{1,t+u}$, conditional on ν_t is given by

$$\begin{aligned}\Pr(\nu_{1,t+u} \leq v | q_t, \omega_t) &= \Pr(h_1^{-1}(e^{\Delta_t}) \leq v | q_t, \omega_t)\\ &= \Pr(e^{\Delta_t} \leq h_1(v) | q_t, \omega_t).\end{aligned}$$

The previous line shows that we shall not need to compute the inverse function $h_1^{-1}(\cdot)$. Coupled with the fact that Δ is Gaussian, this means deriving the cumulative distribution function is straightforward:

$$\begin{aligned}\Pr(e^{\Delta_{t+u}} \leq h_1(v) | q_t, \omega_t) &= \Pr(\Delta_{t+u} \leq \ln h_1(v) | q_t, \omega_t)\\ &= \Pr(q_{t+u} + d_\omega \omega_{t+u} \leq \ln h_1(v) | q_t, \omega_t).\end{aligned}$$

Now observe that

$$\begin{aligned}\Pr(q_{t+u} + d_\omega \omega_{t+u} \leq \ln h_1(v) | q_t, \omega_t) &= E_t[1_{\{q_{t+u} + d_\omega \omega_{t+u} \leq \ln h_1(v)\}}]\\ &= E_t[1_{\{q_{t+u} + d_\omega \omega_{t+u} \leq \ln h_1(v)\}}]\\ &= 1 - E_t[1_{\{q_{t+u} + d_\omega \omega_{t+u} > \ln h_1(v)\}}].\end{aligned}$$

Lemma A3 implies that

$$\begin{aligned}E_t[1_{\{q_{t+u} + d_\omega \omega_{t+u} > \ln h_1(v)\}}] &= \Phi\left(\frac{q_t + \mu_q u + d_\omega[e^{-\lambda_x u}\omega_t + (1 - e^{-\lambda_x u})\bar{\omega}] - \ln h_1(v)}{\sqrt{\sigma_q^2 u + 2 d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x} \sigma_q \sigma_Y + d_\omega^2 \frac{1-e^{-2\lambda_x u}}{2\lambda_x} \sigma_Y^2}}\right)\\ &= \Phi\left(\frac{q_t + d_\omega \omega_t + \mu_q u + d_\omega(1 - e^{-\lambda_x u})(\bar{\omega} - \omega_t) - \ln h_1(v)}{\sqrt{\sigma_q^2 u + 2 d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x} \sigma_q \sigma_Y + d_\omega^2 \frac{1-e^{-2\lambda_x u}}{2\lambda_x} \sigma_Y^2}}\right),\end{aligned}$$

where

$$\begin{aligned}\mu_q &= (\beta_2 - \beta_1) + \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right),\\ \sigma_q &= \sigma_{\xi,1} - \sigma_{\xi,2} + \left(\frac{1}{\psi_2} - \frac{1}{\psi_1}\right)\sigma_Y,\\ d_\omega &= \left(\gamma_2 - \frac{1}{\psi_2}\right) - \left(\gamma_1 - \frac{1}{\psi_1}\right).\end{aligned}$$

Hence,

$$\Pr(\nu_{1,t+u} \leq v | q_t, \omega_t) = \Phi\left(\frac{\ln h_1(v) - [q_t + d_\omega \omega_t + \mu_q u + d_\omega(1 - e^{-\lambda_x u})(\bar{\omega} - \omega_t)]}{\sqrt{\sigma_q^2 u + 2 d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x} \sigma_q \sigma_Y + d_\omega^2 \frac{1-e^{-2\lambda_x u}}{2\lambda_x} \sigma_Y^2}}\right).$$

Because $q_t + d_\omega \omega_t = \ln h_1(\nu_{1,t})$, we have

$$\Pr(\nu_{1,t+u} \leq v | q_t, \omega_t) = \Phi\left(\frac{\ln \frac{h_1(v)}{h_1(\nu_{1,t})} - [\mu_q u + d_\omega(1 - e^{-\lambda_x u})(\bar{\omega} - \omega_t)]}{\sqrt{\sigma_q^2 u + 2 d_\omega \frac{1-e^{-\lambda_x u}}{\lambda_x} \sigma_q \sigma_Y + d_\omega^2 \frac{1-e^{-2\lambda_x u}}{2\lambda_x} \sigma_Y^2}}\right).$$

The density function $p_{\nu_{1,t+u}}(v|q_t,\omega_t)$ is given by

$$p_{\nu_{1,t+u}}(v|q_t,\omega_t)=\frac{d\Pr(\nu_{1,t+u}\le v|q_t,\omega_t)}{dv}$$
$$=\frac{1}{\sqrt{\sigma_q^2 u+2d_\omega\frac{1-e^{-\lambda_x u}}{\lambda_x}\sigma_q\sigma_Y+d_\omega^2\frac{1-e^{-2\lambda_x u}}{2\lambda_x}\sigma_Y^2}}$$
$$\times\phi\left(\frac{\ln\frac{h_1(v)}{h_1(\nu_{1,t})}-[\mu_q u+d_\omega(1-e^{-\lambda_x u})(\bar\omega-\omega_t)]}{\sqrt{\sigma_q^2 u+2d_\omega\frac{1-e^{-\lambda_x u}}{\lambda_x}\sigma_q\sigma_Y+d_\omega^2\frac{1-e^{-2\lambda_x u}}{2\lambda_x}\sigma_Y^2}}\right)\frac{h_1'(v)}{h_1(v)}.$$

Because

$$\frac{h_1'(v)}{h_1(v)}=\frac{\gamma_1\gamma_2}{v(1-v)\mathbf{R}_t(v)},$$

we obtain (A56). When $\mu_q=0$, the limit of (A56) as $u\to\infty$ when $v\in(0,1)$ gives zero. When $v=0$ or 1, the limit is infinite, but symmetry and the fact that $p_{\nu_{1,t+u}}(v|q_t,\omega_t)$ is a probability density function (and hence integrates to one) implies that $\lim_{u\to\infty}p_{\nu_{1,t+u}}(v=0|q_t,\omega_t)=\frac{1}{2}\delta(v)$ and $\lim_{u\to\infty}p_{\nu_{1,t+u}}(v=1|q_t,\omega_t)=\frac{1}{2}\delta(v-1)$.

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