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Asset price bubbles from heterogeneous beliefs about mean reversion rates

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Abstract Harrison and Kreps showed in 1978 how the heterogeneity of investor beliefs can drive speculation, leading the price of an asset to exceed its intrinsic value. By focusing on an extremely simple market model—a finite-state Markov chain—the analysis of Harrison and Kreps achieved great clarity but limited realism. Here we achieve similar clarity with greater realism, by considering an asset whose dividend rate is a mean-reverting stochastic process. Our investors agree on the volatility, but have different beliefs about the mean reversion rate. We determine the minimum equilibrium price explicitly; in addition, we characterize it as the unique classical solution of a certain linear differential equation. Our example shows, in a simple and transparent manner, how heterogeneous beliefs about the mean reversion rate can lead to everlasting speculation and a permanent "price bubble."

Keywords Asset price bubble · Heterogeneous beliefs · Minimal equilibrium price

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1 Introduction

Are the prices of assets in financial markets determined by their intrinsic values? This question has been debated for decades. The US internet stock bubble of 1998–2000 and the market's dramatic collapse in Fall 2008 suggest that prices can be far above

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their intrinsic values even in a very efficient market. The mechanisms underlying such price bubbles have been a continuing focus of attention in the finance and economics literature.

The pioneering paper of Harrison and Kreps [7] was perhaps the first to explain how heterogeneous beliefs can lead to speculation, thereby producing a bubble. The central idea is that an investor who owns an asset today might be able to sell it tomorrow to a *more optimistic* investor at a relatively high price. So heterogeneity—specifically, the possible emergence of more optimistic investors—makes the option to sell the asset valuable. Therefore the price of the asset should be higher than any investor's assessment of its intrinsic value (the price the investor would pay if forced to hold it forever). The paper [7] develops this idea in detail, for an asset with finitely many possible dividend rates and Markovian dynamics.

The most widely used market models involve stochastic differential equations (SDEs), not Markov chains. Therefore it is natural to seek an SDE-based example of the Harrison–Kreps mechanism. The main goal of the present paper is to give such an example.

We consider an asset that entitles its holder to a continuous dividend stream. The dividend rate has mean-reverting dynamics

$$dD = \kappa(\theta - D)\,dt + \sigma\,dw.$$

Our investors agree on the values of σ and θ, but they disagree on the mean reversion rate κ. Notice that when $D < \theta$, the most optimistic investor is the one with the largest value of κ, but when $D > \theta$, the situation is opposite: the most optimistic investor is the one with the smallest value of κ. Since the value of D crosses θ repeatedly, our investors change from optimistic to pessimistic and vice versa. Newly-pessimistic investors sell to newly-optimistic ones, at a price that exceeds either investor's intrinsic value.

Our framework is the continuous-time analogue of Harrison and Kreps'. In particular, our investors are risk-neutral, and short-selling is prohibited. Our main result is an explicit formula for the minimal equilibrium price. We also characterize this price as the unique classical solution (with linear growth at infinity) of a certain second-order, linear differential equation. Finally, we examine its dependence on the parameters of the model.

We are not the first to discuss an SDE version of the Harrison and Kreps' framework. That distinction belongs to Scheinkman and Xiong, who studied how overconfidence can lead to speculative bubbles [11, 12]. Their work is related to ours: they, too, consider an asset that pays continuous dividends, with a mean-reverting dividend rate. There are, however, important differences:

(a) We assume the dividend rate is observable. Scheinkman and Xiong assume it is not: rather, their investors see only "signals," i.e., noisy measurements of the rate.
(b) We assume the investors differ by using distinct choices of the mean reversion rate κ when predicting future dividends. Scheinkman and Xiong assume instead that the investors differ by placing greater confidence in different "signals."
(c) We ignore transaction costs, while Scheinkman and Xiong include them; in particular, they study the degree to which transaction costs influence the size of the bubble.

In brief: our goal is different from that of Scheinkman and Xiong. We discuss a simpler model, focusing on the underlying PDE issues; the result is a transparent, PDE-based version of the Harrison–Kreps framework. Scheinkman and Xiong by contrast discuss a more complex model; the result is a more realistic framework, with broader financial and economic implications.

We mention in passing a related but different theme in the recent literature on market heterogeneity. Can "irrational investors" survive, and can they have significant market impact? Milton Friedman famously argued that the answer is no: an irrational investor will inevitably lose money, thereby being driven from the market [6]. But other authors have questioned this view. For example, DeLong et al. [4] discuss a model in which an irrational investor can survive in the long run—and can even earn a higher return than a rational investor. Kogan et al. [8] use a finite-horizon lognormal model to demonstrate that survival and price impact are independent issues: though the wealth of an irrational investor may quickly become relatively small, he may still have substantial price impact. Yan [13] takes a different approach, defining a "survival index" and arguing that the investors with the lowest value of the index will survive in the long run.

The viewpoint of the present paper is different from those of the articles just cited. We do not distinguish between "rational" and "irrational" investors, and we do not examine the long-term experience of any investor. However, our model is consistent with the hypothesis that one of our two investors is rational (i.e., his model for the evolution of the dividend rate is correct). With this interpretation, the model provides a mechanism by which the presence of irrational investors can lead to speculation, raising the market prices of assets to levels well above their intrinsic values.

2 The model

As mentioned in the introduction, we present a continuous-time version of the Harrison–Kreps framework. Two groups of investors compete for ownership of the dividend stream of an asset. Each believes the dividend rate D is mean-reverting, but they have different beliefs about the mean reversion rate: group i uses the model

$$dD = \kappa_i(\theta - D)\,dt + \sigma\,dw, \tag{2.1}$$

where $i = 1, 2$, and $\kappa_1 > \kappa_2 > 0$. Note that while the models used by the two groups are different, the associated measures on path space are absolutely continuous by Girsanov's theorem. Thus the two groups have the same view about which paths are "possible"; they simply differ about which are more likely.

Our investors are risk-neutral; thus, they assess the attractiveness of the asset by considering the expected present value of its dividends (while it is held) plus the amount realized by its sale (if it is sold). Each group of investors has unlimited resources, and short-selling is prohibited. Therefore the asset will be held, at any particular time, by the group that assigns it the higher value. One could consider more than two groups of investors; but there is no reason to do so, because (as will become clear in due course) only the groups with the lowest and highest values of κ would ever own the asset.

We shall explain the notion of an "equilibrium price," demonstrate the existence of the minimal equilibrium price, and provide an explicit formula for it. More specifically:

- In this section we briefly summarize the Harrison–Kreps framework, explaining in particular the notion of an "equilibrium price."
- In Sect. 3 we discuss the minimal equilibrium price, showing in particular that it is strictly higher than the asset's intrinsic value.
- In Sect. 4 we offer a PDE characterization of the minimal equilibrium price. Specifically, we suggest that it is the unique classical solution (with linear growth at infinity) of a certain linear differential equation. This solution can be written explicitly in terms of *parabolic cylinder functions*. We also show in Sect. 4 that group 1 holds the asset when $D(t) < \theta$, while group 2 holds it when $D(t) > \theta$. This confirms the intuition offered in the Introduction that the asset is always held by the more optimistic group.
- In Sect. 5 we prove that the solution of the PDE considered in Sect. 4 is indeed the minimal equilibrium price discussed in Sect. 3. Our main tool is a comparison result from the theory of viscosity solutions.
- In Sect. 6 we discuss the equilibrium price and its dependence on the parameters of our model.

Before defining an equilibrium price, we must discuss the "intrinsic value" of the asset. As already mentioned, our investors are risk-neutral. To keep the model simple, we assume they discount future income at the same constant rate λ. So at time t, a member of group i calculates that the present value of the asset if held forever is

$$E^{Q_i}\left\{\int_t^\infty e^{-\lambda(s-t)} D(s)\,ds \,\middle|\, D(t)\right\},$$

where Q_i is the measure associated with the process (2.1), $i = 1, 2$. The asset will be held by the group assigning it a higher value, so we define its intrinsic value to be

$$\max_{i=1,2} E^{Q_i}\left\{\int_t^\infty e^{-\lambda(s-t)} D(s)\,ds \,\middle|\, D(t)\right\}.$$

Since $D(t)$ is a stationary process, the asset's intrinsic value depends only on the value of D at time t. Thus the intrinsic value at time t is $I(D(t))$, where

$$I(D) = \max_{i=1,2} E^{Q_i}\left\{\int_0^\infty e^{-\lambda s} D(s)\,ds \,\middle|\, D(0) = D\right\}.$$

Since the evolution of D is very simple, the function I is easy to evaluate. An investor of type i calculates that

$$D(t) = \theta + e^{-\kappa_i t}\big(D(0) - \theta\big) + \sigma \int_0^t e^{\kappa_i(s-t)}\,dw(s). \tag{2.2}$$

So $E^{Q_i}(D(t)) = \theta + e^{-\kappa_i t}(D(0) - \theta)$, and

$$E^{Q_i}\left\{\int_0^\infty e^{-\lambda s} D(s)\,ds\right\} = \int_0^\infty e^{-\lambda s} E^{Q_i}\big(D(s)\big)\,ds = \frac{\theta}{\lambda} + \frac{1}{\lambda+\kappa_i}\big(D(0) - \theta\big).$$

Maximizing over $i = 1, 2$ we get

$$I(D) = \begin{cases} \frac{\theta}{\lambda} + \frac{1}{\lambda+\kappa_1}(D-\theta) = \frac{D}{\lambda+\kappa_1} + \frac{\theta\kappa_1}{\lambda(\lambda+\kappa_1)}, & \text{if } D \le \theta, \\ \frac{\theta}{\lambda} + \frac{1}{\lambda+\kappa_2}(D-\theta) = \frac{D}{\lambda+\kappa_2} + \frac{\theta\kappa_2}{\lambda(\lambda+\kappa_2)}, & \text{if } D \ge \theta. \end{cases} \tag{2.3}$$

Notice that the intrinsic value can be negative, if D is sufficiently negative.

We turn now to the notion of an equilibrium price. Suppose at time t the dividend rate is $D(t)$ and the price of the asset is $P(D(t), t)$. Assume furthermore that group i holds the asset. They could sell it at any stopping time $\tau \ge t$, and the present value of the resulting income stream would be

$$\sup_{\tau \ge t} E^{Q_i}\left\{ \int_t^\tau e^{-\lambda(s-t)} D(s)\,ds + e^{-\lambda(\tau-t)} P\big(D(\tau), \tau\big) \,\middle|\, D(t) \right\}.$$

At equilibrium, for the group that holds the asset, this should be equal to $P(D(t), t)$. Therefore we make the following definition.

Definition 2.1 An *equilibrium price* is a continuous function $P(D, t)$, defined for $D \in \mathbb{R}$ and $t \ge 0$, such that

$$P(D, t) \ge I(D)$$

and

$$P(D, t) = \max_{i=1,2} \sup_{\tau \ge t} E^{Q_i}\left\{ \int_t^\tau e^{-\lambda(s-t)} D(s)\,ds + e^{-\lambda(\tau-t)} P\big(D(\tau), \tau\big) \,\middle|\, D(t) = D \right\},$$

where τ ranges over nonnegative stopping times.

As stated in Harrison and Kreps [7], this is a partial-equilibrium version of Radner's approach [10]. In particular, we are assuming that the investors' views about the future are common knowledge.

Our simple mean-reverting model for the dividend rate $D(t)$ has the drawback that D can become negative. This issue is familiar: many widely-used interest rate models (for example the Vasiček and Hull–White models) have the same feature. By choosing suitable parameters, we can arrange that the probability of D being negative is relatively small. Note that in the presence of carrying costs, negative dividend rates may be realistic.

Since D can be negative, so can the intrinsic value $I(D)$. We shall show in due course that the minimum equilibrium price can also be negative; in fact, as $D \to -\infty$, it approaches $I(D)$. It might seem odd that we permit the price of an asset to be negative. But this is consistent with the Harrison–Kreps framework, which *requires* that the asset be held by one of the two groups of investors. This framework ignores the possibility that the investors may have other, more attractive investment opportunities.

We are assuming that although the existence of diverse beliefs is common knowledge, each group stays firm in its choice of a market model. This assumption seems reasonable, since the mean reversion rate is notoriously difficult to estimate from historical data (much more difficult than σ or θ). It is, however, quite different from the framework used in [11, 12] or [5], where investors adjust their models using signals from the market and some sort of filtering.

A central issue in the economics of speculation is whether the resulting bubble must eventually burst. The mechanism considered here produces a bubble that is permanent, in the sense that it persists as long as the model remains valid. However, if the investors change their estimates of κ, the bubble could disappear.

3 The minimal equilibrium price

We now give a characterization of the minimal equilibrium price. We show in the process that the minimal equilibrium price is independent of time.

Theorem 3.1 *Define a sequence* $P_k(D,t)$, $k=0,1,\dots,$ *as follows*: $P_0(D,t)=I(D)$, *and for each* $k=2,3,\dots,$

$$P_k(D,t)=\max_{i=1,2}\sup_{\tau\geq t}E^{Q_i}\left\{\int_t^\tau e^{-\lambda(s-t)}D(s)\,ds+e^{-\lambda(\tau-t)}P_{k-1}\big(D(\tau),\tau\big)\,\middle|\, D(t)=D\right\}.$$

Then $P_k(D,t)$ *is nondecreasing in* k. *The limit*

$$P_*(D,t)=\lim_{k\to\infty}P_k(D,t)$$

is time-independent and is the minimal equilibrium price.

Proof Because $P_0(D,t)$ does not depend on time t and $D(t)$ is a stationary process, $P_k(D,t)$ is independent of t for every k; thus $P_k(D,t)=P_k(D)$, where

$$P_k(D)=\max_{i=1,2}\sup_{\tau\geq 0}E^{Q_i}\left\{\int_0^\tau e^{-\lambda s}D(s)\,ds+e^{-\lambda\tau}P_{k-1}\big(D(\tau)\big)\,\middle|\,D(0)=D\right\}. \quad (3.1)$$

Since $\tau=0$ is a stopping time, we have $P_k(D)\geq P_{k-1}(D)$, so $\{P_k(D)\}_{k\geq 1}$ is a nondecreasing series. Therefore

$$P_*(D)=\lim_{k\to\infty}P_k(D)$$

exists. Passing to the limit in (3.1), we see that P_* is an equilibrium price.

To show that this is the minimal equilibrium price, consider any equilibrium price $Q(D,t)$. It suffices to show that $Q(D,t)\geq P_k(D)$ for all k, and we do this by induction. The initial step is clear, since $Q(D,t)\geq I(D)=P_0(D)$ from the very definition of an equilibrium price. For the inductive step, suppose $Q(D,t) \geq P_{k-1}(D)$. Then

$$\begin{aligned} Q(D,t) &\geq \max_{i=1,2} \sup_{\tau \geq t} E^{Q_i} \left\{ \int_t^\tau e^{-\lambda(s-t)} D(s)\, ds + e^{-\lambda(\tau-t)} P_{k-1}\big(D(\tau)\big) \,\Big|\, D(t) = D \right\} \\ &= P_k(D) \end{aligned}$$

by combining the definition of an equilibrium price with the inductive hypothesis. The induction is now complete, and we conclude in the limit $k \to \infty$ that

$$Q(D,t) \geq P_*(D).$$

Thus $P_*(D)$ is the minimal equilibrium price. □

It is natural to ask whether the minimal equilibrium price has a speculative bubble. In other words, is $P_*(D)$ strictly larger than $I(D)$? The answer is yes. In fact, even P_1 is strictly larger than $I(D)$. To see this, consider the choice $\tau = 1$ in the definition of P_1. This gives

$$P_1(D) \geq \max_{i=1,2} E^{Q_i} \left\{ \int_0^1 e^{-\lambda s} D(s)\, ds + e^{-\lambda} I\big(D(1)\big) \,\Big|\, D(0) = D \right\}. \tag{3.2}$$

Taking the viewpoint of group i, and always conditioning on $D(0) = D$, we have

$$E^{Q_i} I\big(D(1)\big) > E^{Q_i} \left\{ \frac{\theta}{\lambda} + \frac{1}{\lambda + \kappa_i} \big(D(1) - \theta\big) \right\}$$

using (2.3). We emphasize that the inequality is strict, because both events $\{D(1) < \theta\}$ and $\{D(1) > \theta\}$ have positive probability. Recalling the derivation of (2.3), we conclude that

$$E^{Q_i} \left\{ e^{-\lambda} I\big(D(1)\big) \right\} > E^{Q_i} \left\{ \int_1^\infty e^{-\lambda s} D(s)\, ds \right\}.$$

Combining this with (3.2), we get

$$P_1(D) > \max_{i=1,2} E^{Q_i} \left\{ \int_0^\infty e^{-\lambda s} D(s)\, ds \,\Big|\, D(0) = D \right\} = I(D).$$

Thus $P_1(D)$ is strictly larger than $I(D)$, as asserted.

The preceding calculation reveals rather clearly the origin of the speculative bubble. At equilibrium, each investor is willing to bid more than the expected value of the future dividend stream to buy the asset. The reason is that if the investor holds the asset until time 1, there is a positive probability that the other group will offer a higher bid than his valuation of the future dividend stream. Such speculative behavior keeps pushing up the asset price until it reaches equilibrium.

There are other equilibrium pricing functions besides P_*. For example, for any positive value of the constant c, $P(D,t) = P_*(D) + ce^{\lambda t}$ is an equilibrium price. This pricing function is like a Ponzi scheme. We take the view, following Harrison and Kreps, that the minimal equilibrium price is the one of practical interest.

4 A differential equation for the minimal equilibrium price

In this section, we provide a differential equation characterization of the minimal equilibrium price. More specifically, we introduce a differential equation and show that there is a unique C^2 solution with linear growth at infinity. Moreover, this unique solution is an equilibrium price. Later, in Sect. 5, we shall show that it is the *minimal* equilibrium price. At the end of this section, we demonstrate that the asset is always held by the more optimistic group of investors.

Theorem 4.1 *Consider the differential equation*

$$\max\{\kappa_1(\theta - D), \kappa_2(\theta - D)\}\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi + D = 0. \tag{4.1}$$

(a) *There is a unique C^2 solution Φ of (4.1) with the property that $\Phi(D) = O(D)$ as $D \to \pm\infty$. Moreover, Φ has the explicit form*

$$\Phi(D) = \begin{cases} C_1 F_{-\upsilon_1}\left(\frac{\theta - D}{\sigma/\sqrt{2\kappa_1}}\right) + \frac{D}{\lambda+\kappa_1} + \frac{\theta\kappa_1}{\lambda(\lambda+\kappa_1)}, & \text{if } D \le \theta, \\ C_2 F_{-\upsilon_2}\left(\frac{D - \theta}{\sigma/\sqrt{2\kappa_2}}\right) + \frac{D}{\lambda+\kappa_2} + \frac{\theta\kappa_2}{\lambda(\lambda+\kappa_2)}, & \text{if } D \ge \theta, \end{cases} \tag{4.2}$$

where C_1 and C_2 are positive constants. Here we assume as usual that $\kappa_1 > \kappa_2$; we have set $\upsilon_i = \frac{\lambda}{\kappa_i}$; and $e^{-w^2/4}F_{-\upsilon_i}(w)$ are called parabolic cylinder functions or Weber–Hermite functions. (For information about $F_{-\upsilon_i}$, see for example Borodin and Salminen [1], A 2.9, p. 639.)

(b) *The solution Φ has linear growth as $D \to \pm\infty$; in fact,*

$$\Phi = \frac{D}{\lambda + \kappa_1} + \frac{\theta\kappa_1}{\lambda(\lambda + \kappa_1)} + o(1) \quad \text{as } D \to -\infty, \text{ and}$$

$$\Phi = \frac{D}{\lambda + \kappa_2} + \frac{\theta\kappa_2}{\lambda(\lambda + \kappa_2)} + o(1) \quad \text{as } D \to \infty.$$

Also, Φ is convex and increasing, with $\frac{1}{\lambda+\kappa_1} < \Phi'(D) < \frac{1}{\lambda+\kappa_2}$ for all D.

(c) *The solution Φ is an equilibrium price.*

Proof We begin by solving the PDE, i.e., proving (a). The discussion is parallel to that of Levendorskiĭ [9] in his analysis of a perpetual option on a mean-reverting underlying. When $D \le \theta$, (4.1) reduces to

$$\kappa_1(\theta - D)\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi + D = 0.$$

Defining

$$\Psi(D) = \Phi(D) - \frac{D}{\lambda + \kappa_1} - \frac{\theta\kappa_1}{\lambda(\lambda + \kappa_1)},$$

we see that $\Psi(D)$ satisfies

$$\kappa_1(\theta - D)\Psi' + \frac{1}{2}\sigma^2\Psi'' - \lambda\Psi = 0. \tag{4.3}$$

Setting $\hat{\sigma} = \sigma/\sqrt{2\kappa_1}$, $w = (D-\theta)/\hat{\sigma}$, and $\Psi(D) = e^{w^2/4}\psi(w)$, (4.3) becomes

$$\left(\lambda - \kappa_1\partial_w^2 + \kappa_1 w\partial_w\right)e^{w^2/4}\psi(w) = 0. \tag{4.4}$$

Letting $\upsilon_1 = \lambda/\kappa_1$, and noticing that $e^{-w^2/4}\partial_w e^{w^2/4}\psi(w) = (\partial_w + w/2)\psi(w)$, we divide (4.4) by $-\kappa_1 e^{w^2/4}$ to get

$$\left(\partial_w^2 + \frac{1}{2} - \upsilon_1 - \frac{w^2}{4}\right)\psi(w) = 0. \tag{4.5}$$

Equation (4.5) is called the *Weber differential equation*. Notice that if $\psi(w)$ is a solution, then $\psi(-w)$ is also a solution. The general solution has the form

$$\psi = A_1 e^{-w^2/4}F_{-\upsilon_1}(-w) + B_1 e^{-w^2/4}F_{-\upsilon_1}(w),$$

where $e^{-w^2/4}F_{-\upsilon_i}(\pm w)$ are called *Weber–Hermite functions*. Undoing the changes of variables, this gives

$$\Psi(D) = A_1 F_{-\upsilon_1}\left(\frac{\theta - D}{\sigma/\sqrt{2\kappa_1}}\right) + B_1 F_{-\upsilon_1}\left(\frac{D-\theta}{\sigma/\sqrt{2\kappa_1}}\right).$$

The Weber–Hermite functions have a series representation (see the Appendix). For numerical evaluation, however, it is more convenient to use the software package Mathematica, specifically its routine *ParabolicCylinderD*$[-\upsilon, \pm w]$.

Renaming the constants and working with Φ rather than Ψ, we have shown that for $D < \theta$ the general solution of (4.1) is

$$\Phi(D) = C_1 F_{-\upsilon_1}\left(\frac{\theta - D}{\sigma/\sqrt{2\kappa_1}}\right) + c_1 F_{-\upsilon_1}\left(\frac{D-\theta}{\sigma/\sqrt{2\kappa_1}}\right) + \frac{D}{\lambda+\kappa_1} + \frac{\theta\kappa_1}{\lambda(\lambda+\kappa_1)}.$$

Similarly, for $D > \theta$ the general solution is

$$\Phi(D) = c_2 F_{-\upsilon_2}\left(\frac{\theta - D}{\sigma/\sqrt{2\kappa_2}}\right) + C_2 F_{-\upsilon_2}\left(\frac{D-\theta}{\sigma/\sqrt{2\kappa_2}}\right) + \frac{D}{\lambda+\kappa_2} + \frac{\theta\kappa_2}{\lambda(\lambda+\kappa_2)}.$$

The asymptotics of $F_{-\upsilon_i}(w)$ as $w \to \pm\infty$ are discussed for example in [2] (see particularly Eqs. (5a) and (5b) on p. 92 and Eq. (25) on p. 40). As $w \to +\infty$, we have

$$F_{-\upsilon_i}(w) = w^{-\upsilon_i}\left(1 + O\left(w^{-2}\right)\right), \tag{4.6}$$

$$F_{-\upsilon_i}(-w) = \frac{\sqrt{2\pi}}{\Gamma(\upsilon_i)}e^{w^2/2}|w|^{\upsilon_i - 1}\left(1 + O\left(w^{-2}\right)\right). \tag{4.7}$$

The four constants c_1, c_2, C_1, C_2 are determined by the conditions that

(i) Φ has at most linear growth as $D \to \pm\infty$, and
(ii) Φ and Φ' are continuous at $D = \theta$.

By (4.7), (i) requires that $c_1 = c_2 = 0$. To get C_1 and C_2 explicitly, we use the relation

$$F'_{-\upsilon_i}(w) = -\upsilon_i F_{-\upsilon_i - 1}(w)$$

(see e.g. [1], p. 639). After a bit of algebra, we find that the continuity of Φ and Φ' at $D = \theta$ is equivalent to

$$C_1 F_{-\upsilon_1}(0) + \frac{\theta}{\lambda + \kappa_1} + \frac{\theta\kappa_1}{\lambda(\lambda + \kappa_1)} = C_2 F_{-\upsilon_2}(0) + \frac{\theta}{\lambda + \kappa_2} + \frac{\theta\kappa_2}{\lambda(\lambda + \kappa_2)},$$
$$C_1 \frac{\lambda}{\sigma}\sqrt{\frac{2}{\kappa_1}} F_{-\upsilon_1 - 1}(0) + \frac{1}{\lambda + \kappa_1} = -C_2 \frac{\lambda}{\sigma}\sqrt{\frac{2}{\kappa_2}} F_{-\upsilon_2 - 1}(0) + \frac{1}{\lambda + \kappa_2}.$$

This 2×2 linear system determines the values of C_1 and C_2. A more explicit formula, which shows clearly that C_1 and C_2 are positive, is given in the Appendix.

We have almost finished part (a) of the theorem. The only remaining assertion is that Φ'' is continuous at $D = \theta$. This follows from (4.1) (which has now been established away from $D = \theta$). Taking the limit as $D \uparrow \theta$ gives

$$\frac{1}{2}\sigma^2 \Phi''(\theta-) = \lambda\Phi(\theta) - \theta$$

and taking the limit as $D \downarrow \theta$ gives

$$\frac{1}{2}\sigma^2 \Phi''(\theta+) = \lambda\Phi(\theta) - \theta.$$

Thus Φ'' is continuous at θ. (We remark that the third derivative is *not* continuous. Its limiting values as D approaches θ from above and below can be found by differentiating (4.1) and then arguing as above.)

Part (b) of the theorem is almost immediate. The linear behavior as $D \to \pm\infty$ is an immediate consequence of (4.6). The convexity of Φ follows from the convexity of $F_{-\upsilon}$, which is evident from the representation

$$F_{-\upsilon}(w) = \frac{1}{\Gamma(\upsilon)} \int_0^\infty t^{\upsilon - 1} e^{-\frac{1}{2}t^2 - wt}\, dt$$

(see e.g. Eq. (36b) on p. 44 of [2]). The final assertion of part (b) that

$$\frac{1}{\lambda + \kappa_1} < \Phi'(D) < \frac{1}{\lambda + \kappa_2}$$

follows immediately from the convexity of Φ combined with its asymptotic behavior as $D \to \pm\infty$.

We turn to part (c) of the theorem, which asserts that the function Φ just determined is an equilibrium price. Taking the viewpoint of group i, and applying Itô's formula, we have

$$
\begin{aligned}
d\big(e^{-\lambda t}\Phi\big(D(t)\big)\big) &= -\lambda e^{-\lambda t}\Phi\,dt + e^{-\lambda t}\Phi'\,dD + \frac{1}{2}e^{-\lambda t}\Phi''(dD)^2 \\
&= e^{-\lambda t}\left[\kappa_i(\theta - D)\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi\right]dt + \sigma e^{-\lambda t}\Phi'\,dw.
\end{aligned}
$$

Since $|\Phi'|$ is bounded,

$$
E^{Q_i}\left(\int_0^\infty e^{-\lambda t}\Phi'\,dw\right)^2 \le C\int_0^\infty e^{-2\lambda t}\,dt < \infty;
$$

it follows that for every stopping time $\tau \ge 0$, $E^{Q_i}(\int_0^\tau e^{-\lambda t}\Phi'\,dw) = 0$. Therefore

$$
\begin{aligned}
E^{Q_i}\left\{e^{-\lambda\tau}\Phi\big(D(\tau)\big)\right\} &= \Phi(D_0) + E^{Q_i}\int_0^\tau e^{-\lambda t}\left[\kappa_i(\theta - D)\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi\right]dt \\
&\le \Phi(D_0) + E^{Q_i}\int_0^\tau e^{-\lambda t}\big[-D(t)\big]\,dt,
\end{aligned}
$$

using (4.1) for the last step. Thus

$$
\Phi(D) \ge E^{Q_i}\left\{\int_0^\tau e^{-\lambda s}D(s)\,ds + e^{-\lambda\tau}\Phi\big(D(\tau)\big)\,\bigg|\, D(0) = D\right\}.
$$

This is true for any $\tau \ge 0$ and for $i = 1, 2$, so

$$
\Phi(D) \ge \max_{i=1,2}\sup_{\tau\ge 0} E^{Q_i}\left\{\int_0^\tau e^{-\lambda s}D(s)\,ds + e^{-\lambda\tau}\Phi\big(D(\tau)\big)\,\bigg|\, D(0) = D\right\}. \tag{4.8}
$$

Taking $\tau = N \to \infty$ we see that

$$
\Phi(D) \ge \max_{i=1,2} E^{Q_i}\left\{\int_0^\infty e^{-\lambda s}D(s)\,ds\right\} = I(D).
$$

Moreover, when we take $\tau = 0$ in the expression on the right-hand side of (4.8) we get $\Phi(D)$. Therefore the inequality is actually an equality, i.e.,

$$
\Phi(D) = \max_{i=1,2}\sup_{\tau\ge 0} E^{Q_i}\left\{\int_0^\tau e^{-\lambda s}D(s)\,ds + e^{-\lambda\tau}\Phi\big(D(\tau)\big)\,\bigg|\, D(0) = D\right\}.
$$

Thus $\Phi(D)$ is an equilibrium price. (The argument also shows that the choice $\tau = 0$ is optimal.) □

We asserted in the introduction that the asset is always held by the more optimistic group. We now prove this statement, using the fact (which will be shown in the next section) that Φ is actually the minimal equilibrium price. Given any $D_0 \ne \theta$, let

$$
\tau_\theta = \inf\big\{t > 0 : D(t) = \theta\big\},
$$

and note that $\tau_\theta > 0$. Let us assume for the moment that $D_0 < \theta$. Then for any $0 < t < \tau_\theta$ we have

$$\max\{\kappa_1(\theta - D(t)), \kappa_2(\theta - D(t))\} = \kappa_1(\theta - D(t)). \tag{4.9}$$

Therefore

$$\begin{aligned} &E^{Q_1}\{e^{-\lambda\tau_\theta}\Phi(D(\tau_\theta))\} \\ &\quad = \Phi(D_0) + E^{Q_1}\int_0^{\tau_\theta} e^{-\lambda t}\left[\kappa_1(\theta - D)\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi\right]dt, \\ &\quad = \Phi(D_0) + E^{Q_1}\int_0^{\tau_\theta} e^{-\lambda t}[-D(t)]\,dt, \end{aligned}$$

using (4.1), (4.9) and the positivity of Φ' for the last step. Thus

$$\Phi(D) = E^{Q_1}\left\{\int_0^{\tau_\theta} e^{-\lambda s}D(s)\,ds + e^{-\lambda\tau_\theta}\Phi(D(\tau_\theta))\,\middle|\, D(0) = D\right\}. \tag{4.10}$$

On the other hand, for any $0 < t < \tau_\theta$ we also have

$$\max\{\kappa_1(\theta - D(t)), \kappa_2(\theta - D(t))\} > \kappa_2(\theta - D(t)). \tag{4.11}$$

Therefore for any stopping time $\tau > 0$,

$$\begin{aligned} E^{Q_2}\{e^{-\lambda\tau}\Phi(D(\tau))\} &= \Phi(D_0) + E^{Q_2}\int_0^{\tau} e^{-\lambda t}\left[\kappa_2(\theta - D)\Phi' + \frac{1}{2}\sigma^2\Phi'' - \lambda\Phi\right]dt \\ &< \Phi(D_0) + E^{Q_2}\int_0^{\tau} e^{-\lambda t}[-D(t)]\,dt, \end{aligned}$$

using (4.1) and (4.11) for the last step. Thus

$$\Phi(D) > E^{Q_2}\left\{\int_0^{\tau} e^{-\lambda s}D(s)\,ds + e^{-\lambda\tau}\Phi(D(\tau))\,\middle|\, D(0) = D\right\}. \tag{4.12}$$

From (4.10) and (4.12) we see that when $D < \theta$, the asset is owned by the investors in group 1. This is consistent with the fact that group 2 is more pessimistic since $\kappa_1 > \kappa_2$. A symmetric argument shows that when $D > \theta$, the asset is owned by the investors in group 2. Evidently, trading occurs precisely when the dividend rate D crosses θ, as the two groups exchange ownership of the asset.

5 Identification of Φ as the minimal equilibrium price

In the last section we identified a particular equilibrium price $\Phi(D)$ by solving the differential equation (4.1). It is natural to ask whether $\Phi(D)$ is somehow special within the family of equilibrium prices. The answer is yes: we show in this section that it is the *minimal* equilibrium price.

Our argument uses a comparison result from the theory of “viscosity solutions.” Roughly, the idea is that the definition of an equilibrium price resembles a stochastic optimal control problem. So we expect P_* to be a viscosity solution of the associated differential equation. But Φ is also a viscosity solution, and the viscosity solution is unique, so $P_* = \Phi$.

The preceding assertions are true, but the efficient argument proceeds a little differently. Since we already know that $P_* \le \Phi$, it suffices to prove the opposite inequality. We do this by showing that P_* is a viscosity supersolution. Since Φ is a viscosity subsolution, a standard comparison theorem gives $P_* \ge \Phi$.

We begin by reminding the reader about the definition of a viscosity solution of (4.1). To match the standard convention, we change the sign of the equation to make the coefficient of second-order term negative; thus we consider viscosity solutions of

$$-\max\bigl\{\kappa_1(\theta - D), \kappa_2(\theta - D)\bigr\}\Phi' - \frac{1}{2}\sigma^2\Phi'' + \lambda\Phi - D = 0. \tag{5.1}$$

Definition 5.1 A function $\mu(D)$ is a *viscosity subsolution* of (5.1) if it is upper semicontinuous and for any $\psi \in C^2$ and any local minimum point d of $\psi - \mu$, we have

$$-\max\bigl\{\kappa_1(\theta - d), \kappa_2(\theta - d)\bigr\}\psi' - \frac{1}{2}\sigma^2\psi'' + \lambda\mu - d \le 0.$$

Similarly, $\mu(D)$ is a *viscosity supersolution* of (5.1) if it is lower semicontinuous and for any $\psi \in C^2$ and any local maximum point d of $\psi - \mu$, we have

$$-\max\bigl\{\kappa_1(\theta - d), \kappa_2(\theta - d)\bigr\}\psi' - \frac{1}{2}\sigma^2\psi'' + \lambda\mu - d \ge 0.$$

Finally, $\mu(D)$ is a *viscosity solution* of (5.1) if it is both a viscosity subsolution and a viscosity supersolution.

The argument summarized informally at the beginning of this section is realized by the following theorem.

Theorem 5.1 *The equilibrium price $\Phi(D)$ identified in Sect.* 4 *and the minimal equilibrium price $P_*(D)$ discussed in Sect.* 3 *have the following properties*:

(a) $P_*(D) \le \Phi(D)$;
(b) $\Phi(D)$ *is an upper semicontinuous function*;
(c) $P_*(D)$ *is a lower semicontinuous function*;
(d) $\Phi(D)$ *is a viscosity subsolution of* (5.1);
(e) $P_*(D)$ *is a viscosity supersolution of* (5.1);
(f) P_* *and* Φ *grow at most linearly, i.e., they satisfy*

$$\bigl|P_*(D)\bigr| + \bigl|\Phi(D)\bigr| \le A|D| + B$$

for all $D \in \mathbb{R}$, where A and B are suitable constants.

Furthermore, conditions (b)–(f) *imply*

$$\Phi(D) \le P_*(D).$$

So $\Phi = P_*$. Thus, the unique classical solution of the differential equation with linear growth at infinity is in fact the minimal equilibrium price.

Proof Part (a) is obvious, since Φ is an equilibrium price. Part (b) is also obvious since Φ is a C^2 function. Part (d) follows from the general result that a classical solution of (5.1) is automatically a viscosity solution.

We now prove (c). Our task is to show that for any sequence $D_j \to D$ we have $P_*(D) \le \liminf_{j\to\infty} P_*(D_j)$. Our main tool is the fact that

$$P_*(D) = \max_{i=1,2} \sup_{\tau\ge 0} E^{Q_i}\left\{ \int_0^\tau e^{-\lambda s} D(s)\, ds + e^{-\lambda\tau} P_*\big(D(\tau)\big) \,\middle|\, D(0) = D \right\}$$

since P_* is an equilibrium price. Fixing the sequence D_j and its limit D, let τ_j be the first time the process reaches $D(t) = D$, starting from $D(0) = D_j$. Evidently

$$P_*(D_j) \ge E^Q\left\{ \int_0^{\tau_j} e^{-\lambda s} D(s)\, ds + e^{-\lambda\tau_j} P_*(D) \right\}$$

for both groups, i.e., for both $Q = Q_1$ and $Q = Q_2$. Thus

$$P_*(D) \le \frac{P_*(D_j)}{E^Q\{e^{-\lambda\tau_j}\}} - \frac{E^Q\{\int_0^{\tau_j} e^{-\lambda s} D(s)\, ds\}}{E^Q\{e^{-\lambda\tau_j}\}}. \tag{5.2}$$

We want to pass to the limit in (5.2). Since $D_j \to D$, we have $\tau_j \to 0$ almost surely. Therefore (by the dominated convergence theorem) the denominators have

$$\lim_{j\to\infty} E^Q\{e^{-\lambda\tau_j}\} = 1.$$

As for the numerator of the far right term, it is easy to see from (2.2) that

$$E^Q\left\{ \int_0^\infty e^{-\lambda s} D(s)\, ds \right\} < \infty.$$

Therefore (using dominated convergence again) we have

$$\lim_{j\to\infty} E^Q\left\{ \int_0^{\tau_j} e^{-\lambda s} D(s)\, ds \right\} = 0.$$

Thus we can pass to the limit in (5.2), getting

$$P_*(D) \le \liminf_{j\to\infty} P_*(D_j)$$

as desired.

Next we prove (e), arguing by contradiction. If P_* is not a viscosity supersolution then there exist a C^2 function ψ and a real number D_0 such that D_0 is a local maximum of $\psi - P_*$, $\psi(D_0) = P_*(D_0)$, and

$$-\max\{\kappa_1(\theta - D_0), \kappa_2(\theta - D_0)\}\psi' - \frac{1}{2}\sigma^2\psi'' + \lambda\psi - D_0 \leq -\delta$$

for some strictly positive constant δ. Let us assume for now that $D_0 < \theta$. (The cases $D_0 > \theta$ and $D_0 = \theta$ will be addressed later.) Then by taking ϵ small enough we can find an interval $[D_0 - \epsilon, D_0 + \epsilon]$ with $D_0 + \epsilon < \theta$ such that for any $D \in [D_0 - \epsilon, D_0 + \epsilon]$,

$$\max\{\kappa_1(\theta - D), \kappa_2(\theta - D)\}\psi' + \frac{1}{2}\sigma^2\psi'' - \lambda\psi + D \geq \frac{\delta}{2} > 0. \tag{5.3}$$

Choosing ϵ smaller if necessary, we can also arrange that

$$\psi - P_* \leq 0 \quad \text{on } [D_0 - \epsilon, D_0 + \epsilon]. \tag{5.4}$$

Let T be the first time $D(s)$ reaches an endpoint of this interval, starting from $D(0) = D_0$, i.e.,

$$T = \inf\{s > 0 : D(s) = D_0 - \epsilon \text{ or } D_0 + \epsilon\}.$$

This stopping time has the property that

$$T > 0 \quad \text{with probability 1}; \tag{5.5}$$

moreover, using (5.3) and the assumption that $D_0 + \epsilon < \theta$,

$$\kappa_1(\theta - D(t))\psi' + \frac{1}{2}\sigma^2\psi'' - \lambda\psi + D(t) \geq \frac{\delta}{2} > 0 \quad \text{for } 0 \leq t \leq T. \tag{5.6}$$

From Itô's formula, we have

$$\begin{aligned} d\big(e^{-\lambda t}\psi(D(t))\big) &= -\lambda e^{-\lambda t}\psi\, dt + e^{-\lambda t}\psi'\, dD + \frac{1}{2}e^{-\lambda t}\psi''\,(dD)^2 \\ &= e^{-\lambda t}\left[\kappa_1(\theta - D)\psi' + \frac{1}{2}\sigma^2\psi'' - \lambda\psi\right]dt + \sigma e^{-\lambda t}\psi'\, dw. \end{aligned}$$

Since $|\psi'(D(t))|$ is uniformly bounded in $[0, T]$, from investor 1's perspective we have

$$E^{Q_1}\left(\int_0^T e^{-\lambda t}\psi'\, dw\right)^2 \leq C\int_0^\infty e^{-2\lambda t}\, dt < \infty,$$

whence $E^{Q_1}(\int_0^T e^{-\lambda t}\psi'\, dw) = 0$. Therefore

$$E^{Q_1}\{e^{-\lambda T}\psi(D(T))\} = \psi(D_0) + E^{Q_1}\int_0^T e^{-\lambda t}\left[\kappa_1(\theta - D)\psi' + \frac{1}{2}\sigma^2\psi'' - \lambda\psi\right]dt$$
$$\geq \psi(D_0) + E^{Q_1}\int_0^T e^{-\lambda t}\left(\frac{\delta}{2} - D(t)\right)dt,$$

using (5.6). Thus

$$\psi(D_0) \leq E^{Q_1}\left\{\int_0^T e^{-\lambda s}D(s)\,ds + e^{-\lambda T}\psi(D(T))\right\} - \frac{\delta}{2}E^{Q_1}\int_0^T e^{-\lambda t}\,dt.$$

On the other hand, since P_* is an equilibrium price, we have

$$P_*(D_0) \geq E^{Q_1}\left\{\int_0^T e^{-\lambda s}D(s)\,ds + e^{-\lambda T}P_*(D(T))\right\}.$$

Combining these inequalities with the relation $\psi(D_0) = P_*(D_0)$, we conclude that

$$E^{Q_1}\{e^{-\lambda T}[\psi(D(T)) - P_*(D(T))]\} \geq \frac{\delta}{2}E^{Q_1}\int_0^T e^{-\lambda t}\,dt.$$

The right-hand side is strictly positive, by (5.5); but the left-hand side is less than or equal to zero by (5.4). This is the desired contradiction.

The case $D_0 > \theta$ is handled similarly. Of course, ϵ must then be chosen so that $D_0 - \epsilon > \theta$. The argument is otherwise unchanged, except that κ_1 gets replaced by κ_2 since $\max_{i=1,2}\{\kappa_i(\theta - D)\} = \kappa_2(\theta - D)$ for $D > \theta$, and the probability measure Q_1 is replaced by Q_2.

The case $D_0 = \theta$ is only slightly different. The interval $[D_0 - \epsilon, D_0 + \epsilon]$ used for the previous cases must be replaced by an interval $[D_1, D_2]$ containing $D_0 = \theta$, such that for any $D \in [D_1, D_2]$,

$$\left|\max\{\kappa_1(\theta - D), \kappa_2(\theta - D)\}\psi'\right| \leq \frac{\delta}{6},$$
$$\max\{\kappa_1(\theta - D), \kappa_2(\theta - D)\}\psi' + \frac{1}{2}\sigma^2\psi'' - \lambda\psi + D \geq \frac{5\delta}{6}, \quad \text{and}$$
$$\psi(D) \leq P_*(D).$$

The first two inequalities imply

$$\frac{1}{2}\sigma^2\psi'' - \lambda\psi + D \geq \frac{2\delta}{3} > 0.$$

We argue as before, using $T = \inf\{s > 0 : D(s) = D_1 \text{ or } D_2\}$. Clearly $T > 0$ with probability 1, and Itô's formula gives

$$E^{Q}\{e^{-\lambda T}\psi(D(T))\} \geq \psi(D_0) + E^{Q}\int_0^T e^{-\lambda t}\left(-\frac{\delta}{6} + \frac{2\delta}{3} - D(t)\right)dt$$
$$= \psi(D_0) + E^{Q}\int_0^T e^{-\lambda t}\left(\frac{\delta}{2} - D(t)\right)dt,$$

whence

$$\psi(D_0) \leq E^Q\left\{\int_0^T e^{-\lambda s} D(s)\,ds + e^{-\lambda T}\psi\big(D(T)\big)\right\} - \frac{\delta}{2}E^Q\int_0^T e^{-\lambda t}\,dt.$$

Using $\psi(D_0) = P_*(D_0)$ and the fact that P_* is an equilibrium price, we conclude as before that

$$E^Q\{e^{-\lambda T}[\psi\big(D(T)\big) - P_*\big(D(T)\big)]\} \geq \frac{\delta}{2}E^Q\int_0^T e^{-\lambda t}\,dt.$$

The right-hand side is strictly positive while the left-hand side is nonpositive, which is again a contradiction. This completes part (e).

The proof of (f) is relatively easy. The fact that Φ has the desired properties was part (b) of Theorem 4.1. As for P_*, recall that $P_* \geq I$ and the intrinsic value I is piecewise linear by (2.3). So P_* is certainly bounded below by a linear function. On the other hand, we already know from part (a) that $P_* \leq \Phi$. So P_* is also bounded above by a linear function, and (f) is complete.

The final assertion of the theorem is that (b)–(f) imply $\Phi(D) \leq P_*(D)$ for all D. This follows from the basic comparison theorem for viscosity super and subsolutions, see e.g. Theorem 5.1 of [3]. □

6 Discussion

We have shown that Φ is the minimal equilibrium price, and that if $\kappa_1 > \kappa_2$ then this market has a bubble, i.e., $\Phi(D)$ is greater than the intrinsic value $I(D)$. As an example, Fig. 1 shows the minimal equilibrium price and the intrinsic value as functions of the initial dividend rate when

$$\kappa_1 = 0.2, \quad \kappa_2 = 0.1, \quad \theta = 0.04, \quad \lambda = 0.02, \quad \sigma = 0.02. \tag{6.1}$$

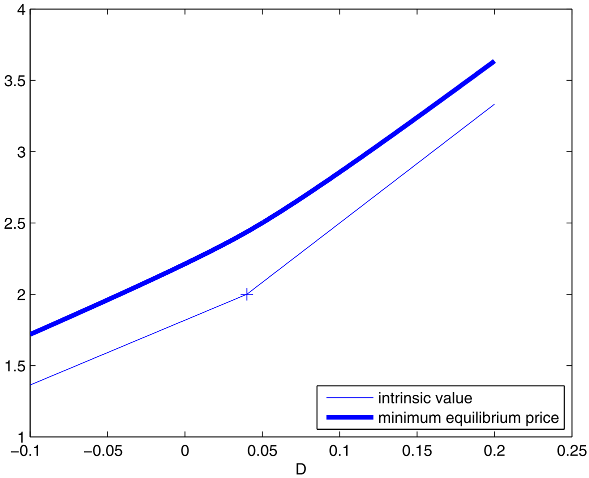


Fig. 1 The minimal equilibrium price (*thick line*) and the intrinsic value (*thin line*) as a function of the initial dividend rate D, when the parameters are given by (6.1)

Here κ_1 and κ_2 differ rather significantly (by a factor of two) and the long-term dividend rate is 4% (a fairly conventional rate). Since the volatility σ is half the size of the long-term rate, $D(t)$ is positive most of the time.

Besides raising the price of the asset, speculative trading also smooths the price of the asset. Indeed, the slope of the intrinsic value $I(D)$ is discontinuous at $D=\theta$, while the minimal equilibrium price is C^2 (though not C^3) at $D=\theta$.

Since we have formulas for Φ and I, our results make it easy to see how the size of the bubble depends on the parameters of the model. In fact, from (2.3) and (4.2), the bubble $B(D)=\Phi(D)-I(D)$ is given by

$$B(D)=\begin{cases} C_1 F_{-\upsilon_1}\left(\frac{\theta-D}{\sigma/\sqrt{2\kappa_1}}\right) & \text{if } D\leq\theta, \\ C_2 F_{-\upsilon_2}\left(\frac{D-\theta}{\sigma/\sqrt{2\kappa_2}}\right) & \text{if } D\geq\theta. \end{cases} \tag{6.2}$$

So the size of the bubble is controlled by the constants C_1 and C_2 in (6.2); the existence of a bubble comes from the positivity of these constants, and the bubble gets larger as C_1, C_2 increase. These constants are given explicitly in the Appendix. The formulas confirm the following very intuitive trends:

(i) The bubble gets larger as $\kappa_1-\kappa_2$ increases with the other parameters held fixed.
(ii) The bubble disappears as $\kappa_1-\kappa_2\to 0$.
(iii) The bubble gets larger as λ decreases with the other parameters held fixed.
(iv) The magnitude of the bubble decreases as $D\to\pm\infty$ with the other parameters held fixed.
(v) The bubble gets larger as σ increases with the other parameters held fixed.

Points (i) and (ii) reflect the fact that our bubble is due to speculation, driven by the heterogeneity of investor beliefs. Point (iii) is natural because the difference $\Phi-I$ reflects the present value of the option to sell in the future; a smaller discount rate makes the present value larger. To explain point (iv), we recall that the group with higher estimate of the mean reversion rate κ owns the asset when $D<\theta$, while the group with lower estimate of κ owns the asset when $D>\theta$. In particular, the investors exchange ownership when $D(t)$ crosses θ. Now, if the initial dividend rate $D(0)$ is far from θ, it will (probably) take a relatively long time for $D(t)$ to reach θ. Since speculative trading is far in the future, its present value is relatively small. The explanation of point (v) is similar: as σ increases, $D(t)$ becomes more volatile, so $D(t)$ crosses θ more often. Since there is more speculative trading, the bubble is larger.

It is natural to consider the *relative* size of the bubble, $R(D)=P_*(D)/I(D)-1$. This is plotted in Fig. 2, using the same parameters as for Fig. 1. Notice that when D is near $\theta=0.04$ (which happens most of the time), the relative size of the bubble is about 20%. Figure 2 also shows that R can be much larger than 20%. Actually, there is even a (negative) value of D where $R=\infty$; this occurs when $I(D)=0$.

Figure 3 explores the dependence of the bubble on the mean reversion rates. The figure displays the size of the bubble at $D=\theta$, as a function of κ_1, when all the other

Fig. 2 The relative magnitude of the bubble, $R(D) = \Phi(D)/I(D) - 1$, when the parameters are given by (6.1)

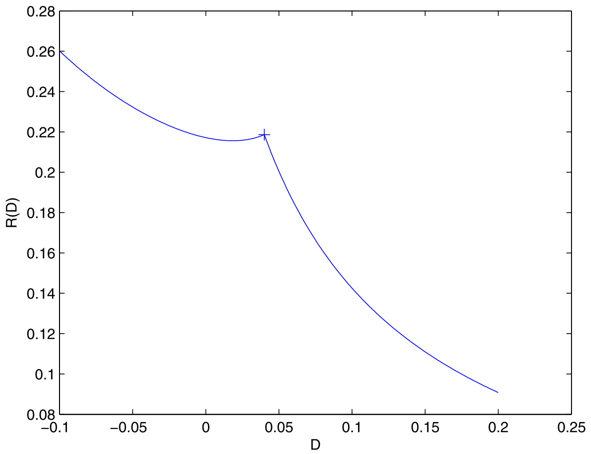
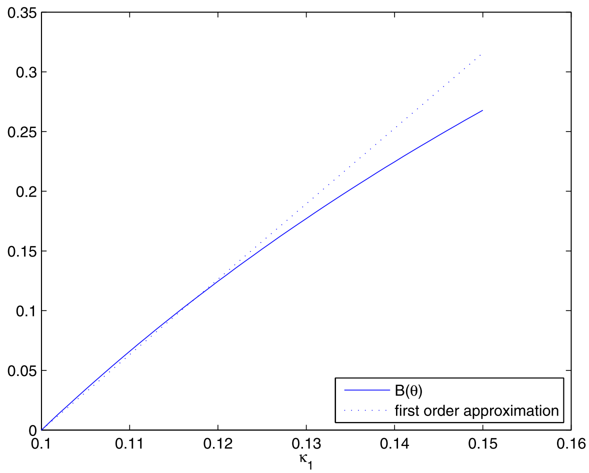


Fig. 3 The bubble $B(\theta) = \Phi(\theta) - I(\theta)$ as a function of κ_1, when the parameters are given by (6.1) except κ_1



parameters are held fixed at the values in (6.1). Notice that the size is approximately linear in $\kappa_1 - \kappa_2$. This is consistent with the fact that C_1 and C_2 are smooth functions of κ_1 and κ_2 which vanish linearly as $\kappa_1 - \kappa_2 \to 0$.

Figure 4 explores the dependence of the bubble on the volatility σ, by reproducing Fig. 1 with different choices of σ. The size of the bubble increases with σ, consistent with observation (v) above.

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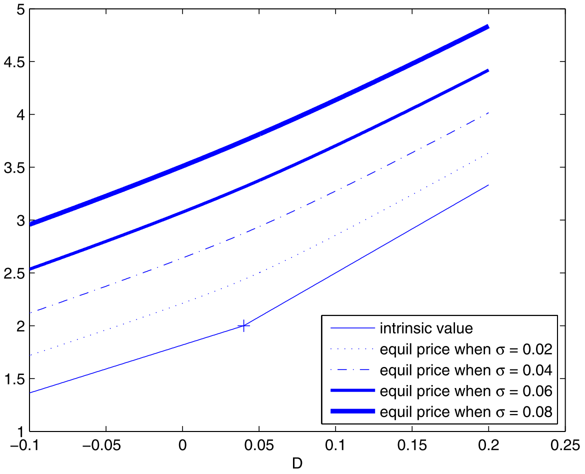

Fig. 4 The minimum equilibrium prices $\Phi(D)$ for different choices of σ, when the parameters are given by (6.1) except σ

Appendix

The following series representation of $F_{-\upsilon_i}(w)$ is given on p. 639 of [1]:

$$F_{-\upsilon_i}(w) = \sqrt{\frac{\pi}{2^{\upsilon_i}}}\left\{\frac{1}{\Gamma\left(\frac{\upsilon_i+1}{2}\right)}\left(1+\sum_{k=1}^{\infty}\frac{\upsilon_i(\upsilon_i+2)\cdots(\upsilon_i+2k-2)}{(2k)!}w^{2k}\right)\right.$$
$$\left. -\frac{\sqrt{2}w}{\Gamma\left(\frac{\upsilon_i}{2}\right)}\left(1+\sum_{k=1}^{\infty}\frac{(\upsilon_i+1)(\upsilon_i+3)\cdots(\upsilon_i+2k-1)}{(2k+1)!}w^{2k}\right)\right\}.$$

In Sect. 4 we gave a 2×2 linear system that determines the constants C_1 and C_2 in the formula (4.2) for Φ. Solving that system and using the series representation just given, one finds that

$$C_1 = \frac{2^{\frac{\upsilon_1}{2}}\sigma(\kappa_1-\kappa_2)}{\sqrt{\pi}\,\Gamma\left(\frac{\upsilon_2+1}{2}\right)\lambda(\lambda+\kappa_1)(\lambda+\kappa_2)}F,$$

$$C_2 = \frac{2^{\frac{\upsilon_2}{2}}\sigma(\kappa_1-\kappa_2)}{\sqrt{\pi}\,\Gamma\left(\frac{\upsilon_1+1}{2}\right)\lambda(\lambda+\kappa_1)(\lambda+\kappa_2)}F,$$

where

$$F = \left(\frac{1}{\sqrt{\kappa_1}\Gamma\left(\frac{\upsilon_1+2}{2}\right)\Gamma\left(\frac{\upsilon_2+1}{2}\right)}+\frac{1}{\sqrt{\kappa_2}\Gamma\left(\frac{\upsilon_1+1}{2}\right)\Gamma\left(\frac{\upsilon_2+2}{2}\right)}\right)^{-1}.$$

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