ECON 2021 - Financial Economics I

Lecture 2 – Arbitrage and Equilibrium

September 17, 2018

Stochastic Discount Factors

- Last time we discussed the "Fund. Eq. of Asset Pricing"

$$P_t = E_t[m_{t+1}X_{t+1}]$$

- We called m_{t+1} a Stochastic Discount Factor (SDF)
- We related m_{t+1} to a utility function.
- Today we study the implications of the mere existence of a SDF, without attempting to relate it to preferences.
- Finance is relatively successful because many of its results are (nearly) preference free. We just need to assume the absence of arbitrage opportunities.

Uses of No Arbitrage Reasoning

There are two leading applications of No Arbitrage reasoning in financial economics:

1. Derivatives Pricing
   - Derivatives payoffs can be *replicated* by dynamically trading other securities. To avoid arbitrage, their prices must equal the value of the replicating portfolio.
2. Term Structure Modeling
   - Prices of riskless bonds of different maturities represent expectations of the *same* SDF over different horizons. This imposes tight restrictions on their yields. Separately modeling bonds of different maturities can easily violate no arbitrage restrictions.

Three Theorems

We start by proving 3 key theorems:

1. Law of One Price $\Leftrightarrow$ Existence of a SDF
2. No Arbitrage $\Leftrightarrow$ Existence of strictly positive SDF
3. Complete Markets $\Leftrightarrow$ SDF is unique

To keep things simple, today we assume just 2-dates and a finite number of states. Later we extend the results to continuous-time with continuous-states.

Four Representations

We also want to understand the meaning and uses of the following alternative representations of the Fund. Eq.

1. $P = \sum_s x(s)q(s)$ } Arrow-Debreu
2. $P = E[mx]$ } SDF
3. $P = \frac{1}{R_f}E^*[x]$ } Risk-Neutral
4. $P = \frac{1}{R_f}E[\eta \cdot x]$ } Equiv. Mart. Measure

where $\pi(s)$ are the agent's (subjective) beliefs about the state probabilities, $1/R_f = \sum_s q(s)$, and

$$m(s) = \frac{q(s)}{\pi(s)} \qquad \pi^*(s) = \frac{q(s)}{\sum q(s)} \qquad \eta(s) = \frac{q(s)/\sum q(s)}{\pi(s)}$$

Comments

1. The AD rep. is the most fundamental. It expresses the idea that assets are bundles of state contingent claims. However, $q(s)$ combines both beliefs and preferences. Sometimes you might want to separate them (e.g, by imposing Rational Expectations).

2. The SDF rep. separates beliefs from preferences. It is the most useful for empirical analysis of equilibrium asset pricing models, at least for those based on RE.

3. The Risk-Neutral rep. is the most useful for theoretical analyses of derivatives markets. It is also useful for computations based on monte carlo simulation, since expected values are easy to simulate.

4. The EMM rep. is useful in models of ambiguity and heterogenous beliefs. For example, η can be interpreted as a distorted probability measure which agents use to construct robust portfolio/savings policies. In continuous-time, Girsanov's Theorem will allow us to construct η very easily.

Classic References

1. Ross (1978) - "A Simple Approach to the Valuation of Risky Streams" *Journal of Business*
2. Harrison & Kreps (1979) - "Martingales and Arbitrage in Multiperiod Securities Markets" *J. of Econ. Theory*
3. Hansen & Richard (1987) - "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models" *Econometrica*

Theorem 1: LOP $\Leftrightarrow$ Existence of a SDF

- The LOP simply says that if 2 assets (or portfolios) have the same payoffs (in every state) then they must have the same price.

- If they don't, then the simplest sort of arbitrage is possible (buy low, sell high).

- It is based on the assumption that investors can costlessly construct (or deconstruct) portfolios of assets.

- Mathematically, the LOP implies $P(x)$ is a linear functional:

$$P(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 P(x_1) + \alpha_2 P(x_2)$$

- That a SDF implies the LOP (LOP $\Leftarrow$ SDF) is trivial. It follows from the linearity of the expectations operator.

- Let $x = y + z$. Then

$$E[mx] = E[m(y+z)] = E[my] + E[mz]$$

- The more interesting half of the theorem is LOP $\Rightarrow$ SDF

- As with many important results, there are many ways to prove this. The most general is based on the Riesz Representation Theorem (bounded linear functionals in a Hilbert space have an inner product representation)

$$P(x) = m \cdot x \equiv E(mx)$$

- However, in finite-dimensional spaces we can be more constructive. I will first provide an algebraic proof, then a geometric proof.

- Algebraic: Suppose there are n assets and s states. Let X by the $n \times s$ payoff matrix. Define the following projection

$$\begin{aligned} m_p &= \text{proj}[m|X] \\ &= X'\theta \qquad \theta \in \mathbb{R}^n \end{aligned}$$

- θ is defined by the following orthogonality condition

$$E[X(m - X'\theta)] = 0 \Rightarrow \theta = E[XX']^{-1}E[Xm] = E[XX']^{-1}P$$

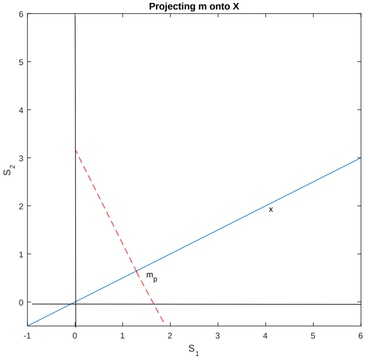
- Therefore, by construction, $m_p = X'E[XX']^{-1}P$ is a valid SDF.

- Although there may be other SDFs, m_p is the unique SDF in X.

- We can visualize this as follows - Suppose there are 2 states, and just 1 asset. Let $x = (4, 2)$, i.e, the asset pays 4 in s_1 and 2 in s_2. Also suppose $\pi_1 = \pi_2 = 1/2$ and $P = 3$.
- An SDF satisfies the equation

$$3 = (1/2)x_1 m_1 + (1/2)x_2 m_2 \qquad \Rightarrow \qquad 3 = 2m_1 + m_2$$

- $m_p = (1.2, 0.6)$ is the unique SDF that lies on the span of X.



- Hansen & Jagannathan (1991) use this idea to derive a lower bound on the volatility of any SDF that prices a given collection of assets.

- Let's consider returns (x/P) rather than payoffs, so the Fund. Eq. takes the form $1 = E(mR)$, where R is an $n \times 1$ vector of returns. Let $\mu = E(R)$ and $\Sigma = \text{cov}(R)$.

- Now project m onto span(R)

$$m = E(m) + \beta'(R - \mu) + \varepsilon \quad \Rightarrow \quad \beta = \Sigma^{-1}\text{cov}(m, R)$$

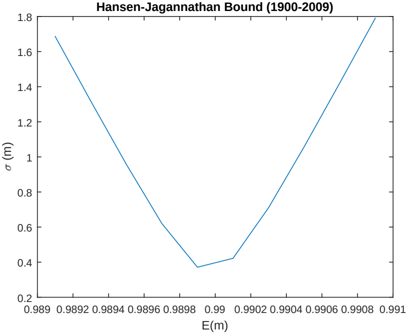
  Since $E(mR) = 1$ we can write $\beta = \Sigma^{-1}[1 - E(m)\mu]$.

- By construction, ε is orthogonal to $\beta'(R - \mu)$. Therefore,

$$\text{var}(m) \geq \beta'\Sigma\beta = [1 - E(m)\mu]'\Sigma^{-1}[1 - E(m)\mu]$$

$$\Rightarrow \sigma_m \geq \sqrt{[1 - E(m)\mu]'\Sigma^{-1}[1 - E(m)\mu]}$$

Here is a plot of the lower bound on σ_m as a function of $E(m)$, using annual returns data on the S&P500 and US T-Bills.



Note that it is very concentrated around the reciprocal of the mean T-Bill return (we do not assume the T-Bill is riskless). This reflects the very low variance of the T-Bill return.

- Geometric Proof: The LOP can be stated as follows

$$\forall(\theta, \hat{\theta}) \quad X'\theta = X'\hat{\theta} \quad \Rightarrow \quad P'\theta = P'\hat{\theta}$$

- This is equivalent to: $X'\theta = 0 \;\; \Rightarrow \;\; P'\theta = 0$.
- Now, $X'\theta = 0$ says that θ is orthogonal to the columns of X, i.e., it is in the 'orthogonal complement' of col(X).
- Thus, $P'\theta = 0$ implies P is orthogonal to the orthogonal complement of col(X).
- But the orthogonal complement of the orthogonal complement of col(X) is just col(X)! That is, P must lie in the space spanned by the columns of X

$$P = Xq$$

which is just the AD rep. of the Fund. Eq.

Theorem 2: No Arbitrage $\Leftrightarrow$ $m > 0$

- We must now be more precise about what we mean by arbitrage. An arbitrage is a portfolio $\theta \in \mathbb{R}^n$ such that either $P'\theta \leq 0$ and $X'\theta > 0$ or $P'\theta < 0$ and $X'\theta \geq 0$.
- For example, an arbitrage is a portfolio strategy that costs you nothing, will definitely not lose money, and *might* make you money.
- Hence, it is defined ex ante rather than ex post.
- Note, the LOP might hold, but there could still be arbitrage opportunities. (*Exercise*: Construct an example).

- That a $m > 0$ implies No Arbitrage (No Arbitrage $\Leftarrow$ $m > 0$) is easy. We know $P'\theta = E[m(\sum \theta_i x_i)]$. Since $\sum \theta_i x_i \geq 0$ for *all* states, and $\sum \theta_i x_i > 0$ for *some* states, then $m > 0$ for all states clearly implies $P'\theta > 0$.
- Hence, the more interesting half of the theorem is that No Arbitrage implies the existence of a strictly positive SDF.
- In finite-dimensional spaces, proofs of this can be based on some version of the Separating Hyperplane Theorem.
- In infinite-dimensional spaces (e.g., with continuous-time trading), technical difficulties can arise, and the theorem is only valid in an approximate sense (discussed next time).

Sketch of Proof:

- Define $M \subset \mathbb{R}^{s+1} = \{(-P(x), x) : x \in \mathbb{X}\}$, where $\mathbb{X}$ is the set of asset payoffs. Given No Arbitrage, the LOP holds and we know M is linear space with $\mathbf{0} \in M$.

- No Arbitrage means that M cannot consist of strictly positive elements (eg., if $x > 0$ then $-P < 0$). Thus, $M \cap \mathbb{R}_{+}^{s+1} = \{0\}$ (ie, M is a hyperplane in $\mathbb{R}^{s+1}$ that only intersects $\mathbb{R}_{+}^{s+1}$ at 0).

- From the Separating Hyperplane Theorem, there exists a linear functional $F : \mathbb{R}^{s+1} \to R$ such that $F(-P, x) = 0 \quad \forall(-P, x) \in M$ and $F(-P, x) > 0 \quad \forall(-P, x) \in \mathbb{R}_{+}^{s+1} \setminus \{0\}$.

- From the Riesz Representation Theorem, we know F can be represented by a vector $(1, q)$,

$$F(-P, x) = (1, q) \cdot (-P, x) = -P + q \cdot x$$

  Since $F > 0$ for $x > 0$, we know $q > 0$. Finally, since $F = 0$ $\forall(-P, x) \in M$, we have $P = q \cdot x$.

Comments:

- Just as $P = E(mx)$ is called the 'Fundamental Eq. of Asset Pricing', the result that No Arbitrage $\Leftrightarrow \exists m > 0$ is (again, pompously) called the 'Fundamental *Theorem* of Asset Pricing'.
- This result says nothing about uniqueness. There may be many $m > 0$ that rule out arbitrage.
- The result is stated in terms of AD state prices, but we can convert to SDF form by defining $m(s) = q(s)/\pi(s)$.
- The earlier projection argument $m_p = \text{proj}[m|\mathbb{X}]$ does not guarantee $m_p > 0$. However, if No Arbitrage applies on $\mathbb{X}$ as well as LOP, then m_p can be extended to a strictly positive m. This result bears some resemblance to the 'Anything Goes' theorem of Sonnenschein-Mantel-Debreu. It implies that if $\mathbb{X}$ satisfies the relatively weak (eg, preference free) result of No Arbitrage, then we can interpret the data as if it was generated by a complete markets general equilibrium economy.

Theorem 3: Complete Markets $\Leftrightarrow$ m is Unique

- It is best to think of complete markets in terms of outcomes, rather than by mechanically counting the number of assets and states.

- In our 2-date/finite-state world, the market is complete if any payoff $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{s}}$ can be attained by some portfolio $\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{n}}$ (ignoring costs and budget constraints)

$$\boldsymbol{w} = \boldsymbol{X}'\boldsymbol{\theta}$$

- This is a system of s equations in n unknowns. Clearly, a necessary condition for complete markets is then $n \geq s$.

- Hence, if payoffs are continuously distributed (e.g, Normal) then we need an *infinite* number of securities to complete the market. However, later we shall see that with dynamic trading, markets can be complete with a small number of assets even when payoffs are continuous, as long as information arrival is itself continuous in a certain sense.

Proof:

- Suppose markets are complete, so that rank$(X') = s$. By the LOP we know $P = Xq$ for some (potentially nonunique) q. Premultiply both sides by $X' \Rightarrow X'P = X'Xq$. Since rank$(X') = s$, $X'X$ is invertible, and so $q = (X'X)^{-1}X'P$ is the unique AD state price vector (and $m = q/\pi$ is the unique SDF).

- Note, if $n = s$ then X and X' are themselves invertible, so $q = X^{-1}P$.

- Now suppose there is a unique q such that $P = Xq$. Note, this requires $n \geq s$. Without loss of generality, assume $n = s$. (If $n > s$ we can eliminate redundant assets). Therefore X' is invertible, which then implies that for any $w \in \mathbb{R}^s$ we can construct the supporting portfolio $\theta = (X')^{-1}w$. Hence, markets are complete.

SUMMARY

- If these 3 theorems were only valid in a 2-date/finite-state world, they wouldn't be very useful. Fortunately, they are much more general.
- However, with many periods, and especially in continuous-time, portfolio strategies can be quite complex, so we need to impose some (technical) restrictions to rule out arbitrage.
- The key idea will be to relate absence of arbitrage to the martingale behavior of appropriately scaled wealth and price processes. The scale factor will be an SDF *stochastic process*. Doob's Optional Stopping Theorem then assures us of the absence of arbitrage.
- In practice, the result that $P = Xq$ is used as follows: We observe prices of traded securities, P. We then 'invert' X to find $q = X^{-1}P$. Given q, we can then compute the No Arbitrage price of any derivative asset with payoff y by simple summation: $P_y = \sum y(s)q(s)$.
- However, what if we want to value an asset when its payoffs are *not* spanned by an existing set of assets?

Equilibrium & Efficiency

- So far we have inferred P given q, or q give P. What if we don't know either? For example, what if we want to value a non-redundant asset?
- This is a question of economic equilibrium. To answer it, we must introduce preferences, budget constraints, and market-clearing.
- We would also like to address the welfare implications of financial markets. To what extent do financial markets efficiently allocate risk and investment resources?
- With complete markets, the answers are well known:
  1. Competitive equilibria are Pareto Optimal (1st Welfare Theorem)
  2. Market prices can be determined by the marginal conditions of 'Representative Agent' (2nd Welfare Theorem)

- With complete markets we can just view agents as directly choosing state-contingent consumption subject to state-contingent prices.

- With just 2-dates/S-states, his problem becomes

$$\max_{C_0, C(s)} U(C_0) + \beta \sum_s \pi(s) U(C(s))$$

s.t. $\quad C_0 + \sum_s q(s)C(s)) = y_0 + \sum_s q(s)y(s)$

where $\{y_0, y(s)\}$ are his current and future (state-contingent) endowments.

- Letting λ be the Lagrange multiplier on the budget constraint (and assuming an interior solution), the optimality conditions are:

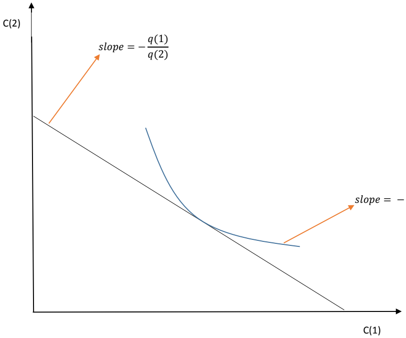
$$\begin{aligned} \lambda &= U'(C_0) \\ \lambda q(s) &= \beta \pi(s) U'(C(s)) \qquad s = 1, 2, \cdots S \end{aligned}$$

- Combining these FOCs gives

$$q(s) = \beta\pi(s)\frac{U'(C(s))}{U'(C_0)} \qquad \Rightarrow \qquad m(s) = \frac{q(s)}{\pi(s)} = \beta\frac{U'(C(s))}{U'(C_0)}$$

Hence, m is related to the Intertemporal Marginal Rate of Substitution

- The beauty of AD General Equilibrium is that it extends basic micro intuition to dynamic/stochastic settings. Portfolio choice can be visualized as follows:



Comments:

- As always, high curvature of an indifference curve reflects a low willingness to substitute. When goods are state-contingent claims, this reflects a high degree of risk aversion.
- Note, if $\frac{q(1)}{q(2)} = \frac{\pi(1)}{\pi(2)}$ then $C(1) = C(2)$. (If AD prices are actuarily fair, then risk aversion implies full insurance).
- From $q(s) = \beta\pi(s)\frac{U'(C(s))}{U'(C_0)}$ we see that AD prices are high when
  1. $\pi(s)$ is high. (State-contingent claims only pay off if the state is realized).
  2. $C(s)$ is low. (Supply & Demand: prices are high when supply is low and demand is low when prices are high).

- Since everyone faces the same prices, the model has strong risk-sharing implications:

Intertemporal

$$\frac{\beta_i \pi_i(s) U'(C_i(s))}{U'(C_{i,0})} = q(s) = \frac{\beta_j \pi_j(s) U'(C_j(s))}{U'(C_{j,0})} \qquad \forall i, j, s$$

$\Rightarrow$ Equality across households of Intertemporal MRS

Across States

$$\frac{U'(C_i(s))}{U'(C_i(s'))} = \frac{q(s)}{q(s')} = \frac{U'(C_j(s))}{U'(C_j(s'))} \qquad \forall i, j, s, s'$$

$\Rightarrow$ Equality across households of MRS state-by-state

Example:

- Suppose households have identical beliefs, rates of time preference, and CRRA utility functions: $U(C) = \frac{1}{1-\gamma}C^{1-\gamma}$. Let $Y_0 = \sum_h y_{h,0}$ and $Y(s) = \sum_h y_h(s)$ be the aggregate endowments in each period. Then the above risk-sharing conditions imply

$$\frac{C_i(s)}{Y(s)} = \frac{C_i(s')}{Y(s)} \quad \text{and} \quad \frac{C_j(s)}{Y(s)} = \frac{C_j(s')}{Y(s)}$$

$\Rightarrow$ Constant shares across states

and

$$\frac{C_{i,0}}{Y_0} = \frac{C_i(s)}{Y(s)} \quad \text{and} \quad \frac{C_{j,0}}{Y_0} = \frac{C_j(s)}{Y(s)}$$

$\Rightarrow$ Constant shares across dates

- By equating supply to demand in both periods we find the market-clearing AD prices: $q(s) = \beta\pi(s)\left(\frac{Y(s)}{Y_0}\right)^{-\gamma}$

- In this example, the equilibrium can be decentralized with a very simple market structure - a simple bond and equity market. Even though S could be very large, we only need 2 assets ($n = 2$) to support the equilibrium (and optimal) allocation!

- This is because we've imposed so many restrictions on preferences, i.e., time-additive CRRA with identical risk aversion coefficients. CRRA implies the willingness to bear (proportional) risk is independent of wealth/endowments, so the allocation of initial wealth is irrelevant to market-clearing equity prices.

- Alternatively, we can impose weaker assumptions on preferences, but stronger assumptions on market structure (eg, full menu of Arrow securities). This allows us to construct a 'Representative Agent', who owns the aggregate endowment, and whose marginal conditions determine market-clearing prices.