# ECON 2021 - FINANCIAL ECONOMICS I

Lecture 5 – Continuous-Time Securities Markets & the Merton Model

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# INTRODUCTION

- We can now apply our results on Ito processes and Dynamic Programming to study continuous-time financial markets. We begin with the classic dynamic portfolio model of Merton (1969,71).
- Merton analyzes a partial equilibrium model of individual lifetime saving and portfolio choice. Asset prices are exogenous, and households confront no non-market risk. Next time we aggregate these optimal policies in order to construct equilibrium prices.
- We will also consider a more modern approach to dynamic portfolio choice called the Martingale Method (Cox and Huang (1989)). This approach first uses stochastic discount factor processes to solve for optimal consumption, and then infers supporting portfolio policies by matching diffusion coefficients.
- But first we revisit some basic issues concerning arbitrage, stochastic discount factors, and equivalent martingale measures.

# Asset Prices

- Suppose there are $N+1$ assets, $N$ risky assets and a riskless asset. The price of the riskless asset obeys the process

$$dR_t = r_t R_t dt \qquad R_0 = 1 \qquad \Rightarrow \qquad R_t = \exp\left[\int_0^t r_s ds\right]$$

  where $r_t$ is the instantaneous riskless rate.

- Prices of the risky assets are governed by the following diffusion processes,

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma dB_t$$

  where $S_t$ is an $N \times 1$ column vector of prices, and $B_t$ is a $k \times 1$ column vector of Brownian motions. For now we assume the drift and diffusion coefficients are constant, so prices are geometric Brownian motions, and returns are i.i.d.

- If the assets pay dividends, then $\frac{dS}{S}$ should be interpreted as a dividend-reinvested rate of return

$$\frac{dS_i}{S_i} = \frac{D_i \cdot dt + dP_i}{P_i}$$

  where the dividend yield $D/P$ is used to purchase new shares, and $P$ represent the ex-dividend price.

- Notice that returns can be correlated, and $\sigma\sigma' \cdot dt$ is the variance-covariance matrix of returns.

# Wealth Dynamics

- For now we ignore consumption and non-market wealth (e.g., labor income). With no outside sources or uses of wealth, wealth only changes due to capital gains or losses.
- Let $Q_t = (R_t, S_t)$ be the vector of asset prices, and $\theta_t$ be the corresponding vector of asset holdings. In discrete-time, the agent's wealth $W_t = \theta_t' Q_t$ evolves according to

$$\theta_{t+\Delta}' Q_{t+\Delta} = \theta_t' Q_t + \theta_t'(Q_{t+\Delta} - Q_t)$$

The continuous-time limit is

$$d(\theta_t' Q_t) = \theta_t' dQ_t \qquad \Rightarrow \qquad W_t = W_0 + \int_0^t \theta_s dQ_s$$

- Since the integral is an Ito integral, meaning $\theta_s$ is evaluated at the left endpoint of the approximating Riemann sum, agents must decide their portfolio allocation without knowing the instantaneous rate of return.
- It is often convenient to consider the riskless asset separately. Letting $\phi_i = \theta_i S_i$ be the value of each risky asset position and $\iota$ be an $N \times 1$ vector of $1$'s, we have

$$dW = Wr \cdot dt + \phi_t'(\mu - r\iota)dt + \phi_t'\sigma dB$$

- It is also often convenient to define $\pi_i = \phi_i / W$ as the share of wealth invested in each risky asset, which then gives

$$\frac{dW}{W} = rdt + \pi'(\mu - r\iota)dt + \pi'\sigma dB$$

# Definitions & Theorems

- The results from Lecture 2 on arbitrage, SDFs, and EMMs extend with minor qualifications to continuous-time trading environments
- **Definition 1**: A self-financing trading strategy is a portfolio process, $\pi_t$, satisfying the budget constraint

$$\frac{dW_t}{W_t} = rdt + \pi_t'(\mu - r\iota)dt + \pi_t'\sigma dB_t$$

  subject to the restriction that $\pi'\sigma \in \mathcal{H}^2$, i.e., $E\left[\int_0^T(\pi_s'\sigma\sigma'\pi_s)ds\right] < \infty$

- **Definition 2**: An arbitrage is a self-financing trading strategy such that $W_0 \leq 0$, $W_T \geq 0$, and either $W_0 < 0$ or $P(W_T > 0) > 0$.
- **Definition 3**: A Stochastic Discount Factor Process, $M$, is a strictly positive Ito process such that $MR$ and $MS$ are martingales.
- **Definition 4**: An Equivalent Martingale Measure is a probability measure, $Q$, such that $S/R$ is a martingale under $Q$, and $Q$ and $P$ share the same null events.
- **Theorem 1**: *An SDF exists if and only if an EMM exists, and the relationship between them is given by* $\xi = M \cdot R$, *where* $\xi$ *is the EMM and* $M$ *is the SDF.*
- **Theorem 2**: *If an SDF (or EMM) exists, and trading strategies are in* $\mathcal{H}^2$ *(or wealth has a lower bound), then there is no arbitrage.*

# Computing SDF Processes

- If we are given price processes, $R$ and $S$, then the martingale properties of $MR$ and $MS$, along with Ito's lemma, can be used to find explicit expressions for the SDF process. Given the SDF (or EMM) we can then proceed to price derivative securities.
- Let $Y = MR$, where $R$ is the riskless asset price, $\frac{dR}{R} = rdt$. Applying Ito's lemma (note the 2nd-order term is 0 here)

$$dY = dM \cdot R + M \cdot dR \qquad \Rightarrow \qquad \frac{dY}{Y} = \frac{dM}{M} + rdt$$

- For $Y$ to be a (local) martingale, its drift must be zero. Therefore, we know the drift of $dM/M$ is $-rdt$. This is the continuous-time counterpart to our previous discrete-time result that $E(m) = 1/R_f$.
- Next, the Martingale Representation Theorem tells us that the diffusion component of $M$ is spanned by the underlying Brownian motions, so we can write

$$\frac{dM}{M} = -rdt - \lambda' dB$$

  for some stochastic process $\lambda$. The $k \times 1$ vector $\lambda$ will turn out to be the market price of risk for each of the underlying Brownian shocks.
- Again applying Ito's lemma to the processes $Y_i = MS_i$ gives

$$dY_i = dM \cdot S_i + M \cdot dS_i + dM dS_i \qquad \Rightarrow \qquad \frac{dY_i}{Y_i} = \frac{dM}{M} + \frac{dS_i}{S_i} + \left(\frac{dM}{M}\right)\left(\frac{dS_i}{S_i}\right)$$

- Since the drift of $dS_i/S_i$ is $\mu_i$, for $Y_i$ to be a (local) martingale we must have

$$(\mu_i - r)dt = -\left(\frac{dM}{M}\right)\left(\frac{dS_i}{S_i}\right)$$

That is, the risk premium on each asset equals its instantaneous covariance with $dM/M$. This is the continuous-time counterpart to our previous discrete-time result, $E(R) - R_f = -R_f \text{cov}(m, R)$.

- Combining these $N$ martingale conditions give us

$$(\mu - r\iota)dt = -\left(\frac{dM}{M}\right)\left(\frac{dS}{S}\right) = \sigma(dB)(dB)'\lambda = \sigma\lambda dt$$

This imposes restrictions on the $\lambda$ process. For example, the solvability of the system $\sigma\lambda = \mu - r\iota$ determines whether markets are complete and a unique SDF exists. If $N < k$ (more Brownian shocks than risky assets), we have more unknowns than equations, and a unique solution (and SDF) will generally fail to exist. However, if $N = k$ and $\sigma$ is nonsingular, we get a *unique* SDF process

$$\boxed{\frac{dM}{M} = -rdt - (\mu - r\iota)'\sigma^{-1}dB}$$

# Computing EMM Processes

- By definition, an Equivalent Martingale Measure is a probability measure, $Q$, that makes $S/R$ a martingale under $Q$.
- Note that an SDF process simultaneously adjusts for time and risk. In contrast, an EMM process only adjusts for risk. You must still discount payoffs using the risk-free rate.
- Constructing an EMM involves taking expectations w.r.t. a new, transformed, probability measure. Changing probability measures is accomplished using Radon-Nikodym derivatives.
- Here's a simple example: Suppose $x \sim N(0,1)$. Its prob measure is

$$dP(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Suppose you want to transform $P$ so that $x \sim N(\mu, 1)$ under $Q$,

$$dQ(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx$$

Note that if we define $\xi(x) = e^{x\mu - \frac{1}{2}\mu^2}$ we have

$$dQ(x) = \xi(x) dP(x) = \frac{1}{\sqrt{2\pi}} e^{x\mu - \frac{1}{2}\mu^2 - \frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx$$

$\xi(x) = \frac{dQ}{dP}$ is an example of a Radon-Nikodym derivative. It transforms a $N(0,1)$ density to a $N(\mu,1)$ density. Note that it is invertible. $\xi(x)^{-1}$ transforms from $N(\mu,1)$ to $N(0,1)$. Also, note $E^P[\xi(x)] = 1$.

- We want a stochastic process analog of this result. This is provided by Girsanov's Theorem.
- Suppose we have a Brownian motion $B_t$ defined relative to a probability measure $P$. Suppose we are also given an $\mathcal{F}_t$-adapted process $X_t$ (remember, $\mathcal{F}_t$ = filtration generated by $B_t$). Now define the process $\xi_t$ as follows

$$\xi_t = e^{\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds} \qquad t \in [0, T]$$

Note that $\xi_0 = 1$.

- To ensure $\xi_t$ is well defined, we impose the restriction that $X_t$ not be 'too volatile'

$$E\left[e^{\int_0^t X_s^2 ds}\right] < \infty \qquad t \in [0, T]$$

This is known as Novikov's condition.

- Using Ito's lemma, the differential of $\xi_t$ is

$$d\xi_t = \xi_t X_t dB_t \qquad \Rightarrow \qquad \xi_t = 1 + \int_0^t \xi_s X_s dB_s$$

Since $\xi_t$ is driftless, it is a local martingale, which satisfies $E^P(\xi_t) = 1$. Given Novikov's condition, we know that it is in fact a global martingale.

- Now, Girsanov's Theorem states that we can use $\xi_t$ as a Radon-Nikodym derivative, and define a new probability measure, $Q$, such that the process

$$d\tilde{B}_t = B_t - \int_0^t X_s ds \qquad \Rightarrow \qquad d\tilde{B}_t = dB - X_t dt$$

  is a Brownian motion relative to $Q$.

- Note that $Q$ effectively adds a drift process, $X_t$, to any diffusion process defined relative to $P$.

$$\begin{aligned} P: \quad dY &= \mu(Y)dt + \sigma(Y)dB \\ Q: \quad dY &= [\mu(Y) + X]dt + \sigma(Y)d\tilde{B} \end{aligned}$$

  The stochastic process $X_t$ plays the same role that $\mu$ did in our previous example, except now it is a drift *process* that is shifted, rather than a single mean value. In finance applications, $X_t$ will be (minus) the equilibrium price of risk.

- The following chain of equalities nicely illustrate the relationships among SDFs, EMMS, and RN derivatives:

$$E^Q[1_A] = \int_A dQ = \int_A \frac{dQ}{dP} dP = \int_A \xi dP = \int_A (M \cdot R) dP = E^P[1_A \cdot MR]$$

- Hence, if $MS$ is a $P$-martingale, then $S/R$ will be a $Q$-martingale. Remembering that $dQ/dP = \xi = M \cdot R$, this can be seen as follows,

$$E^P[MS] = E^Q\left[\frac{dP}{dQ}MS\right] = E^Q\left[\frac{1}{\xi}MS\right] = E^Q\left[\frac{MS}{MR}\right] = E^Q\left[\frac{S}{R}\right]$$

*Comments*:

1. The terminology *Equivalent* Martingale Measure means that $P$ and $Q$ are equivalent in the sense that they share the same null events. If one equals 0, so does the other. The necessity of this can be seen from the definition of a Radon-Nikodym derivative, $dQ/dP$.

2. The Novikov condition guarantees that $\xi$ is not just a local martingale, but a global martingale as well. An analogous restriction must be imposed to ensure that $MS$ is a global martingale. As usual, this takes the form of an $\mathcal{H}^2$ restriction on the price of risk, $E\int \lambda'\lambda dt < \infty$.

3. Note that we restricted the time interval to be finite when defining an EMM. Due to the Law of Large Numbers, it can be very difficult to construct equivalent martingale measures over infinite horizons. Incidentally, the reason we only distort the drift is that in continuous-time, differences in diffusion coefficients can be detected using an arbitrarily short time interval. Hence, for 2 processes to be equivalent, they must have the same diffusion coefficients.

# THE MERTON MODEL

- The Merton model is the springboard for most of modern financial economics. It was the first dynamic model of the interaction between saving and portfolio allocation. It was also the first application of the Ito calculus in economics. The 1969 *ReStat* paper focuses on a couple of special cases. The 1971 *JET* paper generalizes and extends the analysis in various directions.

- The model contains 3 key results:
  1. If asset prices are geometric Brownian motions (returns are i.i.d.), then the classic 'Separation Theorems' of the CAPM apply in an intertemporal setting. Investors can span the Mean-Variance frontier with just 2 mutual funds (or just 1 fund of risky assets if there is a riskless asset).
  2. If prices are geometric Brownian motions and preferences belong to the HARA class (Hyperbolic Absolute Risk Aversion), then decision rules are linear in wealth, and the model can be solved analytically.
  3. If prices are geometric Brownian motions, and preferences have Constant Relative Risk Aversion, then optimal portfolios are constant, even if agents have finite horizons (i.e., the portfolio policy is 'myopic').

# HARA Utility Functions

- The HJB equation of the Merton model is a nonlinear PDE, which generally fails to possess an analytical solution. However, when asset prices are geometric Brownian motions and preferences are HARA, then you can obtain explicit solutions.
- The defining feature of HARA is linear risk tolerance

$$\frac{-U'}{U''} = \beta + \frac{C}{\gamma}$$

HARA utility functions are just the solutions of this ODE.

$$HARA: \qquad U(C) = \alpha \frac{\gamma}{1-\gamma}\left(\beta + \frac{C}{\gamma}\right)^{1-\gamma}$$

- HARA functions nest several workhorse preference specifications as special cases:

$$\lim_{\gamma\to\infty} \Rightarrow U(c) = -\beta e^{-c/\beta} \quad \} \quad CARA$$

$$\beta = 0 \Rightarrow U(C) = \frac{1}{1-\gamma}C^{1-\gamma} \quad \} \quad CRRA$$

$$\gamma = -1 \Rightarrow U(C) = -\frac{1}{2}(\beta - C)^2 \quad \} \quad Quadratic$$

# The Infinite Horizon CRRA Problem

- For simplicity, assume there is a single risky asset, which follows the process: $dS/S = \mu dt + \sigma dB$. The investor's objective function is

$$\max_{c,\pi} E_0 \int_0^\infty e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} dt$$

$$\text{s.t.} \qquad dW = [(r + \pi(\mu - r))W - C]dt + \pi\sigma W dB$$

  Notice that we now subtract the rate of consumption from the wealth accumulation equation.

- Applying Ito's lemma, the stationary HJB equation is

$$\delta V = \max_{c,\pi} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + [(r + \pi(\mu - r))W - C]V_w + \frac{1}{2}\pi^2\sigma^2 W^2 \cdot V_{ww} \right\}$$

- The FOCs are:

$$C: \qquad C^{-\gamma} = V_w \qquad \Rightarrow \qquad C^{1-\gamma} = (V_w)^{1-1/\gamma}$$

$$\pi: \qquad (\mu - r)W \cdot V_w + \pi\sigma^2 W^2 V_{ww} = 0 \qquad \Rightarrow \qquad \pi = \left(\frac{-V_w}{W \cdot V_{ww}}\right)\left(\frac{\mu - r}{\sigma^2}\right)$$

- Guess: $V(W) = AW^{1-\gamma}$. This implies,

$$\begin{aligned} V_w &= (1-\gamma)AV_w \quad &\Rightarrow \quad & C = [(1-\gamma)A]^{-1/\gamma}W \\ V_{ww} &= -\gamma(1-\gamma)AW^{-\gamma-1} \quad &\Rightarrow \quad & \frac{-V_w}{W \cdot V_{ww}} = \frac{1}{\gamma} \end{aligned}$$

- Sub these into the HJB equation and cancel the common $AW^{1-\gamma}$ term,

$$\delta = \frac{1}{1-\gamma}(1-\gamma)^{1-1/\gamma}A^{-1/\gamma} + (1-\gamma)\left\{\left[r + \frac{(\mu-r)^2}{\gamma\sigma^2}\right] - [(1-\gamma)A]^{-1/\gamma} - \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right\}$$

- Now solve for $(1-\gamma)^{-1/\gamma}A^{-1/\gamma}$

$$(1-\gamma)^{-1/\gamma}A^{-1/\gamma} = \frac{1}{\gamma}\left[\delta - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right]$$

- This gives us the policy functions:

$$\begin{aligned} C &= \frac{1}{\gamma}\left\{\delta - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right\}W \\ \pi &= \frac{\mu - r}{\gamma\sigma^2} \end{aligned}$$

*Comments*:

1. Notice the portfolio policy is the same as in the static CAPM with quadratic preferences! The combination of constant, i.i.d. expected returns with preferences that make attitudes toward multiplicative risks independent of your wealth level (ie, CRRA), convert the portfolio choice problem into an essentially static problem. You are basically confronting the same problem over and over again. Still, the solution has a strong intuitive appeal, at least as a first approximation. The risky portfolio share is increasing in the relative rate of return, and decreasing in both the amount of risk and the degree of risk aversion. Also, note that constant portfolio *shares* doesn't mean you never trade. Instead, it produces an apparent 'contrarian' response to price changes - selling stocks that have risen in price, and buying those that have fallen.

2. Notice also that consumption is a fixed fraction of wealth (ie, saving rates are constant). Again, the comparative statics are intuitive,

$$\begin{array}{lll} \delta \uparrow & \Rightarrow & C \uparrow \\ \sigma^2 \uparrow & \Rightarrow & C \downarrow \quad \text{if} \quad \gamma > 1 \end{array}$$

3. In the knife-edge case of log utility ($\gamma = 1$), things simplify dramatically

$$C = \delta W$$

   In this case, the income and substitution effects resulting from changes in expected returns exactly offset each other, and saving rates are independent of the rate of return.

# The Finite Horizon CRRA Problem

- People don't live forever. It is often argued that the life-cyle should (and does) have strong effects on savings and portfolio choice. As a first step toward addressing these issues, we simplify in the opposite direction, by assuming that agents know exactly when they will die. The investor's problem is now

$$\max_{c,\pi} E_0 \int_0^T e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} dt$$

s.t. 1.) $dW = [(r + \pi(\mu - r))W - C]dt + \pi \sigma W dB$

2.) V(T) = 0 (no bequest)

where $T$ is the known lifespan. A positive bequest could easily be incorporated.

- Now $t$ becomes a relevant state variable and the HJB equation becomes a PDE

$$-V_t = \max_{c,\pi} \left\{ e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} + [(r + \pi(\mu - r))W - C]V_w + \frac{1}{2}\pi^2\sigma^2 W^2 \cdot V_{ww} \right\}$$

- The FOC for $\pi$ is the same as before. The FOC for $C$ becomes

$$e^{-\delta t}C^{-\gamma} = V_w \qquad \Rightarrow \qquad C = \left[e^{\delta t}V_w\right]^{-1/\gamma}$$

- Subbing the FOCs back into the HJB gives us a nonlinear PDE. Our only hope is to employ a separation of variables stategy. Let's guess

$$V(t,W) = e^{-\delta t}A(t)^{\gamma}W^{1-\gamma}$$

  Now, instead of having an undetermined coefficient to solve for, we have the undetermined *function* $A(t)$ to solve for. Still, if our guess works, it's a lot simpler to solve an ODE for $A(t)$ than a the original PDE for $V$!

- The functional form in $W$ indeed works, and the $W$ terms cancel out of the HJB. We are left with the following ODE

$$\delta - \gamma A^{-1}\dot{A} = (1-\gamma)^{-1/\gamma}A^{-1} - (1-\gamma)\left\{(1-\gamma)^{-1/\gamma}A^{-1} - \left[r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right]\right\}$$

- Multiplying by $A$ and collecting terms give us

$$\dot{A} = \phi A - (1-\gamma)^{-1/\gamma} \qquad \phi = \frac{1}{\gamma}\left\{\delta - (1-\gamma)\left[r + \frac{(\mu - r)^2}{2\gamma\sigma^2}\right]\right\}$$

- We must solve this (linear) ODE s.t. $A(T) = 0$. One can easily see that the homogeneous solution is of the form $A(t) = Be^{\phi t}$, while the particular solution is $A(t) = \frac{(1-\gamma)^{-1/\gamma}}{\phi}$. Adding the two together and imposing the boundary condition gives $B = -(1-\gamma)^{-1/\gamma}\phi^{-1}e^{-\phi T}$. Therefore,

$$A(t) = \frac{(1-\gamma)^{-1/\gamma}}{\phi}\left[1 - e^{-\phi(T-t)}\right]$$

The implied policy functions are then

$$C = \frac{\phi}{1 - e^{-\phi(T-t)}} W \qquad \pi = \frac{\mu - r}{\gamma\sigma^2}$$

*Comments*:

1. Notice we get the same portfolio policy as before. This is not too surprising given the assumptions. Remember, with i.i.d. returns and CRRA utility, the agent's portfolio problem is essentially static, so the horizon, whether finite or infinite, doesn't matter.

2. Notice, however, that the horizon does matter for the consumption/savings policy. As you would expect, you consume more of your wealth as you get older. This captures the right-hand side of Modigliani's life-cyle savings policy. To get the rising left-hand side we must extend the model to include other state variables, e.g., nontradeable labor income.

# CARA Utility

- With CRRA, a rich person is just a 'scaled-up' version of a poor person. Portfolio allocations are constant, and identical, and consumption is a fixed fraction of wealth.
- With CARA [$U(c) = -\alpha^{-1} e^{-\alpha c}$] the policy functions turn out to be

$$
\begin{aligned}
C &= rW + \left[ \frac{\delta - r + (\mu - r)^2 / 2\sigma^2}{\alpha r} \right] \\
\pi &= \frac{\mu - r}{r \alpha \sigma^2 W}
\end{aligned}
$$

- The agent now invests a fixed dollar amount in the risky asset. His share of wealth in the risky asset declines as his wealth grows. (Because relative risk aversion increases). This portfolio behavior is grossly at odds with the data. That's why people prefer CRRA to CARA.
- However, notice that with CARA, rich people save a higher fraction of their wealth, which is arguably more consistent with the data.
- CARA is often used when shocks are additive (eg., idiosyncratic labor income), since there are no wealth effects in this case. Equilibrium prices remain unaffected by the distribution of wealth.

# Labor Income

- The basic Merton model assumes all income derives from invested wealth. This may be a decent approximation for Donald Trump, but for the rest of us, labor income is very important.
- In principle, it is straightforward to incorporate labor income. Let $y =$ labor income, and assume it follows the process:

$$dy = \lambda(y) \cdot dt + \sigma_y \cdot dB_y$$

The budget constraint then becomes:

$$dW = [(r + \pi(\mu - r))W + \lambda(y) - c]dt + \pi\sigma W dB + \sigma_y dB_y$$

- Now $y$ becomes a state variable, and the HJB equation contains a few additional terms,

$$\delta V = \max_{c,\pi} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + [(r + \pi(\mu - r))W + \lambda(y) - C]V_w + \frac{1}{2}\pi^2\sigma^2 W^2 \cdot V_{ww} \right.$$

$$\left. + \lambda(y) \cdot V_y + \tfrac{1}{2}\sigma_y^2 \cdot V_{yy} + \rho\pi\sigma\sigma_y W V_{yw} \right\}$$

where $\rho =$ the correlation between $y$ and the risky asset return.

- Note, we are following tradition in the finance literature and ignoring the disutility of labor. Labor income is assumed to be exogenous. This is quite restrictive. With endogenous labor supply, agents can adjust labor supply in response to market outcomes (e.g., delay retirement if the market crashes).
- In general, even with exogenous labor income, the problem becomes unsolvable (at least analytically). However, there are a couple of interesting cases to consider.
- Case 1: Labor income is riskless and *tradeable*. Merton (1971) discusses this case. The agent can fully 'capitalize' his labor income and incorporate it into his initial financial wealth. For example, if $y = \bar{y}$ is constant, then 'human capital' is $H = \bar{y}/r$, and this just gets added to $W_0$. The problem then becomes identical to the original one.
- Case 2: Labor income is riskless and *nontradeable*. In practice, people can't fully borrow against their future labor income due to obvious moral hazard considerations. In this case, the agent effectively has a risk-free investment in the form of labor income. He can then simply adjust his financial portfolio in response. For example, let $H =$ Human wealth and $W =$ Financial wealth. Then invest $\pi(W + H)$ dollars in the risky asset, and $(1 - \pi)(W + H) - H$ in the risk-free. Notice that the share of *financial* wealth invested in the risky asset is

$$\hat{\pi} = \frac{\pi(W + H)}{W} = \pi\left(1 + \frac{H}{W}\right)$$

- Note, $H/W$ changes over the lifecycle. It is high when you are young (because $H$ is high and $W$ is low), and low when you are old. Hence, young people should hold more risky assets in their financial portfolios than old people. This is a common piece of investment advice.

# HEDGING

- Of course, the more realistic case is that labor income is both risky and correlated with market returns. The FOC for $\pi$ is

$$\pi = -\left(\frac{V_w}{\sigma^2 W V_{ww}}\right)\left[(\mu - r) + \overbrace{\rho\sigma\sigma_y\left(\frac{V_{wy}}{V_w}\right)}^{\text{hedging demand}}\right]$$

  If your labor income is positively correlated with the market ($\rho > 0$) and more income lowers the marginal value of wealth ($V_{wy} < 0$), then you will want to hedge your labor income risk by investing less in risky tradeable assets. In fact, you might even want to short the risky asset ($\pi < 0$) if the hedging demand offsets the positive risk premium. This perhaps explains the limited participation in financial markets by many individuals.

- Hedging occurs not only in response to labor income, but in response to any state variable that changes the investment problem in some relevant way. Other leading examples include changes in the 'investment opportunity set' (ie, predictable mean returns) or changes in beliefs about returns that might occur in response to learning.

- In some cases, the problem can be reduced to an ODE, which is much easier to solve. A widely studied example is when the drift of asset prices is stochastic (i.e., time-varying expected returns). There is abundant empirical evidence in support of this. In particular, suppose the risky asset follows the process:

$$\frac{dS}{S} = \mu(X)dt + \sigma dB$$

where the (exogenous) state variable $X$ follows its own diffusion process

$$dX = \alpha(X)dt + GdB_x$$

- If preferences are CRRA and the horizon is infinite, one can readily verify that the value function takes the form, $V(X, W) = f(X)W^{1-\gamma}$. Consumption continues to be proportional to wealth, and a common $W^{1-\gamma}$ can be canceled from the HJB equation. We are then left with a 2nd-order ODE in the unknown function $f(X)$.
- The sign of the hedging demand depends on $\gamma$. Assume (w.l.o.g) that increases in $X$ represent favorable changes in the investment opportunity set ($V_x = f'(X)W^{1-\gamma} > 0$). The sign of the hedging demand is the same as the sign of $V_{xw} = (1-\gamma)f'(X)W^{-\gamma}$. Hence, if $\gamma < 1$ an increase in expected returns increases the hedging demand. Relatively risk tolerant investors want to take advantage of a more favorable investment climate. However, if $\gamma > 1$ then the hedging demand is negative. Relatively risk averse investors, with rapidly declining marginal value of wealth, will respond to higher expected returns by scaling back their risky investments.

# The Martingale Method

- Portfolios are a derived demand, i.e., a means toward an end. Presumably, people don't derive utility from asset trading, but instead trade assets in order to manage their ability to acquire and consume the things they really care about.
- Given this, we can study optimal portfolio behavior in 2 distinct steps: (1) Given an SDF process that summarizes the prices of all state- and date-contingent claims, solve the agent's lifetime consumption plan. Importantly, given knowledge of the SDF process, the agent has a single lifetime budget constraint. (2) Figure out which portfolio policies will support this optimal consumption plan.
- In general, there will be many different supporting portfolios. Roughly speaking, the more complex the securities (e.g., nonlinear/state-contingent), the less frequently they will need to be traded.
- This 2-step approach is called the 'Martingale Method' because it views wealth as the price of an asset, an asset which yields consumption as its dividend. Like any asset price, its dividend inclusive price follows a martingale when scaled by the SDF.
- There are 2 advantages to the Mart. Method: (1) It is easier. Rather than having to solve a nonlinear PDE for the value function, a simpler *linear* PDE must be solved, (2) It doesn't rely on an exogenously specified asset market structure. The asset market structure emerges as part of the problem's solution. The main drawback is that it relies on a given SDF process, so is better suited to complete markets environments. With incomplete markets, the DP/HJB equation approach is more natural.

- Rather than describe the method in full generality, I apply it to the simple 2-asset/CRRA example we solved earlier when studying the Merton problem. This will allow us to more easily compare the two approaches.

- The objective function is the same as before,

$$\max_{c} E_0 \int_0^\infty e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} dt$$

  The only difference is that we are now only choosing consumption.

- The key difference concerns the budget constraint. Rather than confronting a sequence of flow budget constraints, the agent has a single lifetime budget constraint.

$$W_0 = E_0 \left( \int_0^\infty M_t C_t dt \right)$$

  where $M_t$ is the SDF process, which obeys the diffusion

$$\frac{dM}{M} = -r dt - \kappa dB$$

  and where $\kappa = (\mu - r)/\sigma$ is the price of risk.

- Letting $\lambda$ be the Lagrange Multiplier on the budget constraint, the FOC is

$$C_t = \left( e^{\delta t} \lambda M_t \right)^{-1/\gamma}$$

- Again, the key idea is that at each moment of time optimally invested wealth finances the future stream of optimally chosen consumption.

$$W_t = E_t\left[\int_t^{\infty} C_s \frac{M_s}{M_t} ds\right]$$

where consumption is valued using the SDF process. Substituting in from the FOC,

$$W_t = M_t^{-1}\lambda^{-1/\gamma}E_t\int_t^{\infty} e^{-(\delta/\gamma)s}M_s^{1-1/\gamma}ds$$

- From Ito's lemma,

$$d(\log M) = -(r+(1/2)\kappa^2)dt-\kappa dB \qquad \Rightarrow \qquad M_s = M_t e^{-(r+.5\kappa^2)(s-t)-\kappa B_{s-t}}$$

Therefore,

$$E_t M_s^{1-1/\gamma} = M_t^{1-1/\gamma}e^{-\phi(s-t)} \qquad \phi = (1-1/\gamma)\left(r+\frac{1}{2\gamma}\kappa^2\right)$$

- Evaluating the integral we find,

$$W_t = \left(\frac{1}{\phi + \delta/\gamma}\right)\lambda^{-1/\gamma} M_t^{-1/\gamma} e^{-(\delta/\gamma)t}$$

  If desired, we could use this expression to compute $\lambda$ as a function of $W_0$ (using the fact that $M_0 = 1$). However, it turns out we don't need it.

- Instead, we use this expression to derive the following diffusion process for optimally managed (log) wealth

$$d(\log W) = -\frac{1}{\gamma} d(\log M) - \frac{\delta}{\gamma} dt = -\frac{1}{\gamma}\left[-r - \frac{1}{2}\kappa^2 + \delta\right] dt + \frac{\kappa}{\gamma} dB$$

- This is the wealth process implied by the optimal consumption process. The idea behind the Martingale Method is to infer the risky portfolio share by matching (by choice of $\pi$) the diffusion coefficient of this process with the diffusion coefficient in the Merton budget constraint. We can then infer the consumption process by matching the drift coefficients.

- The Merton budget constraint can be written

$$d(\log W) = \left[r + \pi(\mu - r) - \frac{C}{W} - \frac{1}{2}\pi^2\sigma^2\right] dt + \pi\sigma dB$$

- Matching the diffusion coefficients we find

$$\pi\sigma = \frac{\kappa}{\gamma} \qquad \Rightarrow \qquad \pi = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}$$

  This is the same portfolio we computed using dynamic programming and HJB equations!

- We could directly find the optimal consumption process by using the FOC along with the given process for $M_t$. We just need to apply Ito's lemma. However, the result would not be in the form of a policy function, relating consumption to wealth. It would instead be the diffusion process obtained by substituting the wealth diffusion process into the policy function. If we want to obtain the $C/W$ ratio directly, we can just match the drift coefficients in the optimal wealth process and the Merton budget constraint,

$$\frac{1}{\gamma}\left[r + \frac{1}{2}\kappa^2 - \delta\right] = \left[r + \pi(\mu - r) - \frac{C}{W} - \frac{1}{2}\pi^2\sigma^2\right]$$

  Plugging in for $\kappa$ and $\pi$, and then solving for $C/W$ we find

$$\frac{C}{W} = \frac{1}{\gamma}\left\{\delta - (1-\gamma)\left[r + \frac{(\mu - r)^2}{2\gamma\sigma^2}\right]\right\}$$

  Again, this is the same as before.