# ECON 2021 - Financial Economics I

## Lecture 6 – The Consumption-Based CAPM Model

October 22, 2018

# INTRODUCTION

- The Merton model of individual portfolio choice is a partial equilibrium model. Asset prices are exogenous.
- The obvious next step is to aggregate, impose market-clearing, and derive equilibrium prices.
- We do this following Lucas (1978) and Breeden (1979). They study endowment economies. Instead of taking prices as exogenous, they take consumption as exogenous. Hence, these models are also partial equilibrium. Still, they are useful in characterizing the equilibrium relationship between consumption and asset prices. If a model's FOCs are violated, it is unlikely that adding additional structure will improve it. Later, we examine how production and endogenous consumption can be incorporated.
- In this lecture we assume throughout that agents have Rational Expectations and their preferences are time-additive, CRRA. Later we extend the analysis to incorporate learning, ambiguity, and recursive preferences.

# A First Step

- Last time we derived the optimal portfolio policy $\pi = \frac{\mu - r}{\gamma\sigma^2}$, where $\pi$ is the share of wealth invested in the risky asset, and $\mu$ is its expected rate of return.

- In equilibrium, *someone* has to hold the risky asset. Likewise, for every borrower there must be a lender. Hence, if the riskless asset is in zero net supply, agents have identical CRRA preferences, and there are no nontradeable sources of wealth, it must be the case that $\pi = 1$ in equilibrium. After imposing this market-clearing condition, we can then interpret the portfolio policy as determining the equilibrium rate of return on the risky asset

$$\mu = r + \gamma\sigma^2$$

- There are a few problems with this strategy:
  1. The derivation assumed that $\mu$ is constant. How do we know this is true?
  2. The Merton model assumes that $\sigma^2$ is both the volatility of invested wealth *and* the volatility of consumption. This is wildly counterfactual.
  3. Where does the risk-free rate, $r$, come from?

- Still, this strategy has some merit. We shall see that if expected consumption growth is constant, and we intepret $\sigma^2$ as the variance of consumption growth, then the above expression for the equilibrium risk premium is valid.

# The Lucas/Breeden Model

- A common misconception is that the CCAPM presumes complete markets and that everyone has the same preferences.

- The CCAPM is really just a statement of an individual's Euler equation. It does not require complete markets or identical preferences. All it really presumes is the ability to freely trade assets in a competitive market. However, given the lack of reliable panel data on individual consumption, tests of the CCAPM often impose these additional assumptions in order to derive restrictions on aggregate consumption data.

- Still, the empirical deficiencies of the complete markets/identical CRRA preferences CCAPM model has motivated most of the work in macro finance during the past few decades, so it's a useful benchmark to start with.

- So suppose the unique SDF process is given by

$$M_t = e^{-\delta t}(C_t/C_0)^{-\gamma}$$

  where $C_t$ is aggregate (per capita) consumption.

- This consumption represents the (nonstorable) 'fruit' produced by a 'tree'. We want to price an equity claim to the dividends, $D$, yielded by this tree. In equilibrium, $C = D$.

- Consumption (and dividends) follow the diffusion process,

$$\frac{dC}{C} = \mu dt + \sigma dB$$

implying that consumption growth is lognormal i.i.d. This is a surprisingly accurate description of aggregate consumption data. However, much recent work explores subtle departures from it (e.g., rare disasters, stochastic volatility, and low frequency drift in $\mu$).

- Applying Ito's lemma to $M_t$ using this process for $C_t$ yields,

$$\frac{dM}{M} = \left[-\delta - \mu\gamma + \frac{1}{2}\sigma^2\gamma(1+\gamma)\right] dt - \sigma\gamma dB$$

- From the previous lecture we know that the riskless rate is the (negative) drift of $dM/M$. Hence,

$$r = \delta + \mu\gamma - \frac{1}{2}\sigma^2\gamma(1+\gamma)$$

- The price of the Lucas tree can be calculated by evaluating the following integral

$$P_0 = E_0 \int_0^\infty M_s D_s ds = D_0 E_0 \int_0^\infty e^{-(r+\frac{1}{2}\gamma^2\sigma^2)s - \gamma\sigma B_s} e^{(\mu - \frac{1}{2}\sigma^2)s + \sigma B_s} ds$$

$$= \frac{D_0}{r + \gamma\sigma^2 - \mu}$$

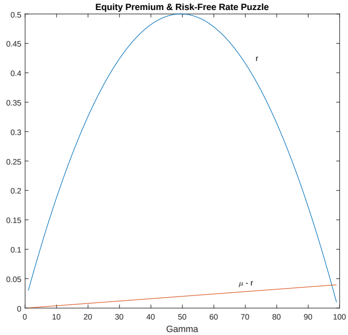
- Therefore, the ex-dividend price follows the same diffusion as $D$ - $dP/P = \mu dt + \sigma dB$, and the dividend reinvested price is given by

$$\frac{dS}{S} = \frac{Ddt + dP}{P} = [r + \gamma\sigma^2 - \mu]dt + \mu dt + \sigma dB = (r + \gamma\sigma^2)dt + \sigma dB$$

- Note that this is consistent with our result from last time,

$$(\mu_S - r)dt = -\left(\frac{dM}{M}\right)\left(\frac{dS}{S}\right) = (\sigma\gamma)(\sigma)dt$$

- The Equity Premium and Risk-Free Rate puzzles refer to the fact that given empirically plausible values of $(\delta, \mu, \sigma^2)$, the model does not generate empirically plausible values of $r$ and/or $\mu_S$ without extremely large values of $\gamma$. The following figure shows this using $\delta = .01$, $\mu = .02$, and $\sigma^2 = (.02)^2$.



- Remember, the mean risk-free rate is about 1%, and the mean Equity Premium is in the range 5-8%. Evidently, we need $\gamma \approx \mathbf{100}$ before the model generates values close to these. At this point, a very strong precautionary saving motive is nearly offsetting a very strong aversion to intertemporal substitution. Such a high degree of risk aversion is not observed in other realms of risk-taking. Either the model is wrong, or there is something different about stock market investing.

- Another way to see the problem is to look at prices, rather than rates of return. The mean $P/D$ ratio has been trending up, due to changes in dividend policy, (that's why Shiller deflates by earnings), but a reasonable value is in the range 20-30. The model's implied $P/D$ ratio is $(r + \gamma\sigma^2 - \mu)^{-1}$. Plugging in $r = .01$, $\mu = .02$, and $\sigma^2 = (.02)^2$ implies that $\gamma \approx (85, 150)$. Even worse, the model predicts the $P/D$ ratio is constant. In the data, $P/D$ ratios exhibit large and persistent fluctuations around the long-run mean. This discrepancy is sometimes called the 'Volatility Puzzle'.

# BUBBLES

- Perhaps the most common explanation of the Volatility Puzzle is to blame bubbles. Bubbles are not (necessarily) irrational. They can be perfectly rational in a world where current outcomes depend on expectations of future outcomes.
- One persuasive counterargument to bubbles is that they rely on an implausible degree of expectations coordination. How do they get started in the first place?
- Models of 'intrinsic bubbles' address this critique. They are based solely on a model's fundamentals. An intrinsic bubble simply represents a solution of the homogeneous part of the HJB equation describing an asset's value.
- Here's a simple example. Suppose dividends follow the usual geometric Brownian motion $\frac{dD}{D} = \mu dt + \sigma dB$. The ex-dividend price satisfies

$$P_t = D_t^\gamma E_t \int_t^\infty e^{-\delta s} D_s^{1-\gamma} ds$$

- Let $V(D)$ be the value of the integral. A recursive representation of $V(D)$ is given by the following HJB eq.

$$\delta V(D) = D^{1-\gamma} + \mu D V'(D) + \frac{1}{2}\sigma^2 D^2 V''(D)$$

- As usual, a solution can be obtained by the method of undetermined coefficients. If we guess $V(D) = AD^{1-\gamma}$, we find that $A = [\delta - \mu(1-\gamma) + \frac{1}{2}\sigma^2\gamma(1-\gamma)]^{-1}$. Substituting out $\delta$ using our previous expression for $r$, we find the same ex-dividend price as before (as we should!)

$$P = \frac{D}{r + \gamma\sigma^2 - \mu}$$

This is called the 'fundamentals solution'. It is the unique solution satisfying the transversality condition, $\lim_{T\to\infty} E_t\left[\frac{M_T}{M_t}P_T\right] = 0$.

- However, if we do not impose the TVC, we can find other solutions. These satisfy the homogeneous equation, $\delta V = \mu D V' + \frac{1}{2}\sigma^2 D^2 V''$. They are called (intrinsic) bubbles. Employing the method of undetermined coefficients, one can verify that in this case they take the form, $V^b(D) = BD^\lambda$, where $\lambda$ is the positive root of the quadratic

$$\frac{1}{2}\sigma^2\lambda(\lambda - 1) + \mu\lambda - \delta = 0$$

- Hence, we get the generalized pricing equation

$$P = \frac{D}{r + \gamma\sigma^2 - \mu} + BD^{\lambda+\gamma}$$

where $B$ is a free parameter. This fits the data better.

# Problems With Rational Bubbles

- Transversality conditions sometimes rule them out (e.g., finite number of infinitely lived agents with complete markets and common knowledge of common priors).
- How do they start?
- Why do they end?
- Trading Volume
- Given these problems, later we develop an alternative theory of bubbles based on heterogeneous beliefs and resale option values.

# Incomplete Markets & Heterogeneity

- Another common suspect in the failure of the complete markets CCAPM is the absence of complete markets. In fact, this was the original conjecture of Mehra & Prescott (1985). It seems promising because individual consumption is more volatile than aggregate consumption, and it's the smoothness of consumption growth relative to stock returns that is the underlying source of the Equity Premium Puzzle.

- However, early attempts to explain the failure with incomplete markets seemed trapped by 2 facts:
  1. Pure idiosyncratic risk won't affect asset prices
  $$\frac{\mu - r}{\sigma_S} = \rho_{\Delta c} \cdot \gamma \sigma_{\Delta c}$$
  and $\sigma_{\Delta c} \uparrow \;\Rightarrow \rho_{\Delta c} \downarrow$.
  2. Unless idiosyncratic income shocks are permanent, individuals can use asset markets to smooth their effects on consumption.

- Constantinides & Duffie (1996) show how persistent idiosyncratic labor income risk can explain the Equity Premium Puzzle.

# Constantinides & Duffie (1996)

- CD work in discrete time. This is actually important. In continuous-time, second moments evolve deterministically, which kills off the CD argument. (See below).
- Suppose there are a large number of agents who receive idiosyncratic labor income shocks. CD construct a process for these shocks such that agents are content to consume their labor income. There is no asset trade, and the Fund. Eq. of asset pricing holds by construction. As a corollary, there is no Equity Premium Puzzle.
- Each agent has preferences, $E\sum_t e^{-\delta t} C_{it}^{1-\gamma}$, $i = 1, 2, \cdots H$. Each agent's labor income (and consumption) follows the process

$$\log\left(\frac{C_{i,t+1}}{C_{i,t}}\right) = \eta_{i,t+1} y_{t+1} - \frac{1}{2} y_{t+1}^2 \qquad \eta_{it} \sim N(0,1)$$

  where $\eta_{it}$ is an idiosyncractic shock, and $y_t$ is the standard deviation of the cross-sectional distribution of consumption growth. Therefore, conditional on $y_{t+1}$, we have

$$\log\left(\frac{C_{i,t+1}}{C_{i,t}}\right) \sim N\left(-\frac{1}{2} y_{t+1}^2, y_{t+1}^2\right)$$

- Note, this implies each agent's consumption follows a martingale: $E(C_{i,t+1}) = C_{it}$

- Now, the essence of the CD model is to reverse engineer a specification for $y_{t+1}$ that satisfies each agent's Euler equation evaluated at his own endowment. It is crucial that the cross-sectional variance depend on the market return. In particular, suppose

$$y_{t+1} = \sqrt{\frac{2}{\gamma(1+\gamma)}} \cdot \sqrt{\delta - \log(R_{t+1})}$$

  Notice, when the market return declines, the cross-sectional variance increases.

- From each agent's Euler eq. we have

$$\begin{aligned}
1 &= E_t\left[e^{-\delta}\left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\gamma} R_{t+1}\right] \\
&= E_t\left[e^{-\delta}\left(e^{\eta_{i,t+1}y_{t+1}-\frac{1}{2}y_{t+1}^2}\right)^{-\gamma} R_{t+1}\right] \\
&= E_t\left[e^{-\delta-\gamma\eta_{i,t+1}y_{t+1}+\frac{1}{2}\gamma y_{t+1}^2+\log(R_{t+1})}\right]
\end{aligned}$$

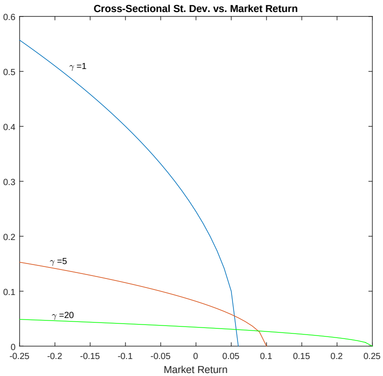
- To evaluate this expectation, we can first condition on $y_{t+1}$

$$E\left[e^{-\gamma\eta_{i,t+1}y_{t+1}}|y_{t+1}\right] = e^{\frac{1}{2}\gamma^2 y_{t+1}^2}$$

- Therefore,

$$\begin{aligned} 1 &= E_t\left[e^{-\delta+\frac{1}{2}\gamma^2 y_{t+1}^2+\frac{1}{2}\gamma y_{t+1}^2+\log(R_{t+1})}\right] \\ &= E_t\left[e^{-\delta+\frac{1}{2}\gamma(1+\gamma)\frac{2}{\gamma(1+\gamma)}(\delta-\log(R_{t+1})+\log(R_{t+1})}\right] = e^0 \end{aligned}$$

- Although the CD model shows how idiosyncratic labor income risk can in principle account for the Equity Premium Puzzle, it still raises empirical questions, e.g., whether its assumptions about the cyclical properties of the cross-sectional distribution of labor income are accurate. It particular, it implies that for modest degrees of risk aversion, the cross-sectional variance has to be strongly countercyclical



# Heterogeneity in Continuous-Time

- A classic paper by Grossman & Shiller (1982) pointed out that the CCAPM does not rely on complete markets, identical preferences, or even identical information sets. However, it also suggested that heterogeneity will not significantly influence risk premia.
- With heterogeneous preferences and incomplete markets each agent has his own SDF process, $M_{it} = e^{-\delta t}[U'(C_{i,t})/U'(C_{i,0})]$.
- If agents can freely trade the same risk free asset, the drift of their $M_{it}$ processes will be identical (and equal to $-rdt$), but their diffusion coefficients will generally differ. To prevent arbitrage, asset prices must be martingales with respect to each agent's SDF process. As usual, this implies

$$(\mu - r)dt = -\left(\frac{dS}{S}\right)\left(\frac{dM_i}{M_i}\right) = \alpha_i\left(\frac{dS}{S}\right)dC_i$$

where $\alpha_i = -U''(C_i)/U'(C_i)$ is agent-$i$'s coefficient of absolute risk aversion. Since the l.h.s. is the same for everyone, notice that agents with relatively high risk aversion have smaller consumption fluctuations. This is accomplished by holding fewer risky assets.

- Dividing both sides by $\alpha_i$ and summing across agents gives

$$\left(\sum_i \frac{1}{\alpha_i}\right)(\mu - r)dt = \left(\frac{dS}{S}\right) dC^a$$

where $C^a = \sum_i C_i$ is aggregate consumption. Note that we can write $\frac{1}{\alpha_i} = \frac{C_i}{\gamma_i}$, where $\gamma_i$ is each agent's coefficient of relative risk aversion. Hence, if we divide both sides by $C^a$ we have

$$\left(\sum_i \omega_{i,t}\gamma_i^{-1}\right)(\mu - r)dt = \left(\frac{dS}{S}\right)\left(\frac{dC^a}{C^a}\right)$$

where $\omega_i = C_i/C^a$ is agent-$i$'s consumption share. This could be time-varying due to uninsured consumption risks.

- This allows us to obtain the following heterogeneous agents CCAPM

$$(\mu - r)dt = \gamma_t^a \left(\frac{dS}{S}\right)\left(\frac{dC^a}{C^a}\right) \qquad \gamma_t^a = \left[\sum_i \omega_{i,t}\gamma_i^{-1}\right]^{-1}$$

Notice that this has the same form as the Lucas/Breeden complete markets CCAPM. The only difference is that the market risk aversion coefficient is the weighted harmonic mean of individual risk aversion, with weights given by consumption shares. Agents with higher consumption shares exert more influence on market risk premia.

- Notice that in contrast to the CD model, the cross-sectional variance of consumption growth does not (directly) influence risk premia. In continuous-time, 2nd-moments evolve deterministically (at least with Gaussian/Brownian motion information structures).

- However, this does not imply heterogeneity is unimportant. Garleanu & Panageas (*JPE*, 2015) recently exploited the GS result, and developed a model in which relative risk aversion differs across agents. They show that risk aversion heterogeneity naturally generates a countercyclical price of risk (with identical CRRA it is constant). Intuitively, market downturns redistribute wealth and consumption from low risk aversion agents (who hold more risky assets) to high risk aversion agents (who are less exposed to risky assets). As a result, $\gamma_t^a$ rises, which increases the market risk premium. Effectively, 'the market' endogenously becomes more risk averse during recessions. (Their model uses recursive preferences, which allows differential heterogeneity in risk aversion and intertemporal substitution.)

# Fat Tails

- The Gaussian density has 'thin tails'. The probability of extreme events goes to zero very fast.

- Brownian motion diffusion models are based on Gaussian innovations. Many have argued that this makes them unsuited to the study of financial markets. It is also the reason why cross-sectional 2nd moments do not play a role in risk premia in standard continuous-time finance models.

- Martin (*RES*, 2013) extends the CCAPM by allowing arbitrary distributions of the innovations to consumption growth.

- Our results linking risk premia to covariances with an SDF process are the same as before, except now we write

$$\frac{dM_t}{M_{t-}} = -r dt + dY_t^c + dY_t^d$$

  where $dY_t^c$ is a continuous local martingale (ie, Brownian motion), and $dY_t^d$ is a purely discontinuous local martingale. The canonical example is a (compensated) Poisson jump process. Every local martingale can be decomposed uniquely in this way. The $t-$ notation emphasizes the fact that $M_t$ is continuous from the right, and possesses a left-hand limit.

- Accounting for the influence of higher-order moments is facilitated by the use of cumulant-generating functions (CGF). Assume (log) consumption growth is an i.i.d. random variable, $g$. The CGF of $g$ is

$$\mathcal{C}(\theta) = \log E \exp(\theta g) = \sum_{n=1}^{\infty} \frac{\kappa_n \theta^n}{n!}$$

where $\kappa_1$ = the mean of $g$, $\kappa_2$ = the variance, $\sigma^2$, of $g$, $\kappa_3/\sigma^3$ = the skewness of $g$, and $\kappa_4/\sigma^4$ = the excess kurtosis. Note that for a Gaussian/Brownian motion diffusion model, only $\kappa_1$ and $\kappa_2$ are nonzero.

- Martin (2013) derives the following expressions for the riskless rate and the equity premium in terms of the CGF of log consumption growth,

$$\begin{aligned} r &= \delta - \mathcal{C}(-\gamma) \\ &= \delta + \mu\gamma - \frac{1}{2}\sigma^2\gamma^2 + \frac{S}{3!}\sigma^3\gamma^3 - \frac{K}{4!}\sigma^4\gamma^4 + h.o.t. \end{aligned}$$

$$\begin{aligned} \mu - r &= \mathcal{C}(1) + \mathcal{C}(-\gamma) - \mathcal{C}(1-\gamma) \\ &= \gamma\sigma^2 + \frac{S}{3!}\sigma^3[1 - \gamma^3 - (1-\gamma)^3] + \frac{K}{4!}\sigma^4[1 + \gamma^4 - (1-\gamma)^4] + h.o.t. \end{aligned}$$

where $\gamma$ is the coefficient of relative risk aversion, $S$ = skewness, and $K$ = excess kurtosis.

- Note that negative skewness (long left tail) and excess kurtosis drive down the riskless rate, since both promote precautionary saving. Note also that if $\gamma > 1$ they increase the equity premium. Unfortunately, unless either $\gamma$ or $(S, K)$ are large, the effects are quantitatively small.

- CGFs are a convenient way to understand the Reitz/Barro disaster risk models. If we suppose disasters have a Poisson arrival rate of $\lambda$, and that conditional on arrival consumption drops are $N(m, s^2)$, then

$$\mathcal{C}(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\left[\exp\left(-\theta m + \frac{1}{2}\theta^2 s^2\right) - 1\right]$$

  This can produce a very convex CGF for 'reasonable' parameter values. For example, if $\delta = .03$, $\mu = .025$, $\sigma = .02$, $\lambda = .017$, $m = .39$, $s = .25$, and $\gamma = 4$, the risk free rate is $1.0\%$ and the equity premium is $5.7\%$.

- Finally, Martin (2013) shows that the results of CD and GS can be reconciled by incorporating both aggregate and idiosyncratic disasters. If idiosyncratic disasters are uncorrelated with aggregate disasters then we are back to the Reitz/Barro world, and idiosyncratic risk doesn't matter. This echoes the results of GS. However, if aggregate and idiosyncratic disasters are correlated, then we recover results like those of CD, and idiosyncratic risk matters.

# THE INTERTEMPORAL CAPM (ICAPM)

- Lucas & Breeden were actually responding to previous efforts by Merton (1973) to endogenize rates of return in his own portfolio model. His ICAPM model is still used as motivation for empirical multifactor asset pricing models.
- Merton based his analysis on the marginal value of wealth, $V_w$, rather than the marginal utility of consumption, $U'(C)$. From the envelope theorem we know,

$$V_w(W, X) = U'(C)$$

So in some sense, the two approaches are equivalent. However, $V_w(W, X)$ will depend on any state variables, $X$, that change the investment opportunity set. In contrast, consumption will react optimally to these anticipated changes, and so will be a sufficient statistic for the state variables. Instead of a multifactor model, we get a *single factor* model. Lucas/Breeden argued that this was a key advantage of the CCAPM.

- To see the relationship between the ICAPM and CCAPM more explicitly, apply Ito's lemma to both sides of the envelope condition, and then divide both sides by $V_w = U'$.

$$-\frac{V_{ww}W}{V_w}\left(\frac{dW}{W}\right) - \frac{V_{wx}X}{V_w}\left(\frac{dX}{X}\right) + O(dt) = -\frac{U''(C)C}{U'(C)}\left(\frac{dC}{C}\right) + O(dt)$$

The Lucas/Breeden model uses the r.h.s to price assets. The Merton ICAPM uses the l.h.s. Clearly, if good consumption data exist, the CCAPM dominates. However, if it easier to measure $X$ and $W$ than $C$, there might be practical advantages to the ICAPM.

# Incorporating Production

- At the outset it was noted that the CCAPM, like the Merton model, is a partial equilibrium model. Consumption is exogenous. However, following Cox, Ingersoll, and Ross (1985), it is straightforward to transform it into a truly general equilibrium model by incorporating a production technology.
- Suppose output is now produced using a 1-sector, stochastic, Constant-Returns-to-Scale/AK production function. That is, consumption and capital are the same good, and capital can be costlessly invested or withdrawn from production, with the rate of return being independent of the quantity invested. Hence, the price of capital is pinned down by the price of the numeraire consumption good. Changes in market capitalization are fully reflected in the quantity of capital, not its price.
- For simplicity, suppose markets are complete, agents have identical log preferences, and there is a single production technology. Equilibrium prices can then be computed by first solving a planner's problem for optimal consumption/capital, and then inferring market-clearing prices using the unique SDF process.

  The planner maximizes $E\int_0^\infty e^{-\delta t}\log(C_t)dt$ subject to the aggregate resource constraint

$$dK = (AK - C)dt + \sigma K dB$$

  In a competitive equilibrium wealth equals capital, $W = K$.

- We've already solved this problem. We know $C = \delta K$, and that equilibrium prices are determined by the SDF process using the usual martingale conditions. For example, the riskfree rate is just (remember $\gamma = 1$ here)

$$rdt = -E\left(\frac{dM}{M}\right) = (\delta + \mu - \sigma^2)dt$$

where $\mu$ is the (endogenous) growth rate of consumption. Given the above consumption function, we know $\mu$ is also the mean growth rate of capital, which is

$$\mu dt = (A - C/K)dt = (A - \delta)dt$$

Hence, the market-clearing interest rate is $r = \delta + (A - \delta) - \sigma^2 = A - \sigma^2$. Again as usual, the equilibrium rate of return on equity claims to the risky technology is the risk-free rate plus the price of risk, which is just $\sigma^2$ with log preferences.

$$\mu = r + \sigma^2 = A$$

- In a sense, the CIR model just reverses what's endogenous and what's exogenous. The assumptions on the production technology completely pin down the rate of return to capital, effectively making asset returns exogenous. Instead, consumption becomes endogenous. Note that CIR does not resolve the Equity Premium Puzzle. The equity premium here is just $\sigma^2$, the variance of consumption growth.

- The model becomes somewhat more interesting if we introduce state variables that shift the investment opportunity set. That is the case CIR focus on.

- For example, suppose productivity depends on an exogenous diffusion process, $A(X)$, where $dX = g(X)dt + \sigma_x dB_x$.

- As discussed last time, this would normally produce a hedging term in the demand function for risky claims. However, with log utility, we know the hedging term disappears. The value function depends on $X$, but it does so separably,

$$V(X, W) = \frac{1}{\delta}\log(W) + f(X)$$

  where $f(X)$ solves a 2nd-order ODE. (See if you can derive it).

- Hence, all that matters is the current level of productivity, $A(X_t)$, and we get

$$\begin{aligned} r_t &= A(X_t) - \sigma^2 \\ \mu_t &= A(X_t) \end{aligned}$$

- CIR use this kind of model to develop a one-factor model of the term structure of interest rates. Given the exogenous dynamics of $X_t$ we can generate the equilibrium dynamics of the short-term interest rate, $r_t$. Given this, we can derive long-term bond yields by evaluating the appropriate expectation. This is especially convenient in continuous-time.

- Note that the additional state variable introduces a new source of risk into the economy. A single equity claim will no longer be sufficient to complete the market. CIR show how adding (zero net supply) derivative securities can complete the market, thus enabling our representative agent solution method.