# ECON 2021 - Financial Economics I

## Lecture 7 – Option Pricing: The Black-Scholes Formula

October 29, 2018

# Background

- An option is the right, but not the obligation, to buy (or sell) an asset under specified terms.
- Options have been around for centuries. In fact, options are implicit in many, if not most, economic decisions.
- However, exchange-traded financial options began only in 1973, at the Chicago Board Options Exchange (CBOE). Not coincidentally, this was the publication date of the Black-Scholes paper.
- Options are an example of a derivative security. The value of a derivative security depends on the value of some underlying asset. (Other examples - futures, swaps, mortgage-backed securities).
- Options exist for many underlying assets, e.g., stocks, bonds, foreign exchange, and commodities.
- No one really knows how big the market is. Recent estimates of the (notional) value of the derivatives market puts it in excess of $500 trillion. (The global stock market is only around $50 trillion.)
- The derivatives market is relatively unregulated (especially the OTC market). Exchange traded options are guaranteed by the exchange. Most derivative positions are 'off balance sheet'.

# Terms & Concepts

- Options are defined by 3 features:
  1. Right to buy or sell.
  2. Strike (or exercise) price.
  3. Expiration date.

- An option to buy is called a call option. An option to sell is called a put option.

- A given stock typically has many different option contracts written on it, differing by call/put, stike price, expiration date.

- A European option can only be exercised at the expiration date. An American option can be exercised at any time before the expiration date.

- The price of the option the premium. For exchange-traded options it is determined by supply & demand. The premium is paid up-front, and is not recovered if the option is not exercised. The seller of an option is said to 'write' the option.

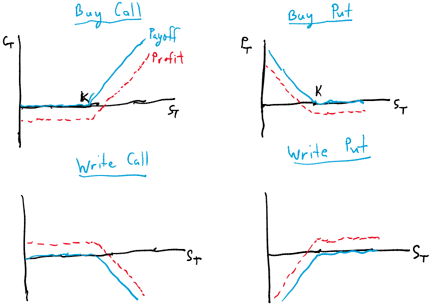
- An option is in-the-money if current exercise yields a profit. Otherwise, it is out-of-the-money. Note, an out-of-the-money option may still have positive value!

# PAYOFF PROFILES

- Let $S$ = stock price, $K$ = Strike Price, $C$ = Value of Call, $P$ = Value of Put

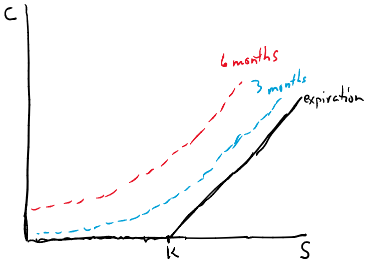
• At expiration date $T$, we have

$\boxed{C_T = \max\{S_T - K, 0\}}$ $\boxed{P_T = \max\{K - S_T, 0\}}$




- Note, the writer of a (naked) call option faces unbounded losses.

- Before expiration, the value of the option will exceed its expiration value.

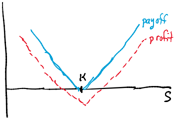



- The Black-Scholes formula provides an equation describing these curves - how they evolve over time, and how they shift in response to changes in model parameters (eg, strike price or volatility).

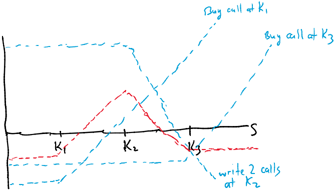
# Portfolios of Options

Part of the reason options are popular is that by combining them into portfolios, you can construct very flexible payoffs. Here are 2 examples:

① Straddle: Buy both a call & put at the same strike price



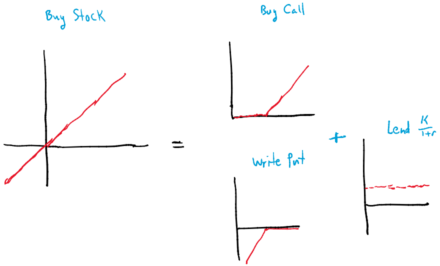

② Butterfly Spread: Buy 2 calls at different strike prices & write 2 calls at an intermediate strike




A straddle is a bet on volatility. It pays off if prices move more than expected (in either direction). A butterfly spread is bet against volatility, which caps your losses.

# Put-Call Parity

- We will focus on European call options. This is not very restrictive, for 2 reasons:
  1. American call options (on non-dividend paying stocks) are never exercised early.
  2. Put-Call Parity: $S = C - P + \frac{K}{1+r}$.




- Buying a call at $K$, writing a put at $K$, and lending $\frac{K}{1+r}$ produces the same payoff at $T$ as just buying the stock. Hence, if we know the call price, we can infer (via no arbitrage) the put price.

# A 1-Period Binomial Formula

- Black-Scholes value options by constructing a payoff replicating portfolio. This gives a no arbitrage valuation for the call option.
- Doing this over arbitrary time intervals takes some mathematics, but for 1-period options, all you need is some algebra.
- Let $S$ = initial stock price, $K$ = Strike Price, and $R = 1 + r$ = Gross interest rate.
- Suppose next period $S$ goes up to $u \cdot S$ with prob $P$, and goes down to $d \cdot S$ with prob $(1 - P)$. We know:

$$
\begin{aligned}
C_u &= \max\{u \cdot S - K, 0\} && \text{value of option if S goes up} \\
C_d &= \max\{d \cdot S - K, 0\} && \text{value of option if S goes down}
\end{aligned}
$$

- Consider forming a portfolio consisting of $x$ dollars worth of stock and $b$ dollars worth of bonds. The value of this portfolio depends on the value of the stock price

$$
\begin{aligned}
&u \cdot x + Rb && \text{if } S \text{ goes up.} \\
&d \cdot x + Rb && \text{if } S \text{ goes down.}
\end{aligned}
$$

- Now pick $x$ and $b$ to replicate the call option payoff

$$\begin{aligned} u \cdot x + Rb &= C_u \\ d \cdot x + Rb &= C_d \end{aligned}$$

Solving for $(x, b)$:

$$x = \frac{C_u - C_d}{u - d} \qquad b = \frac{u \cdot C_d - d \cdot C_u}{R(u - d)}$$

- The current value of the portfolio is then,

$$x + b = \frac{1}{R}\left(\frac{R - d}{u - d} C_u + \frac{u - R}{u - d} C_d\right)$$

By no arbitrage, this must also be the value of the call option.

- Define $q = \frac{R-d}{u-d}$. From no arbitrage, $u > R > d$. Hence, $0 < q < 1$, and we can think of $q$ as a probability and write

$$\boxed{C = \frac{1}{R}\left[q \cdot C_u + (1 - q) \cdot C_d\right] = \frac{1}{R}\hat{E}[C(T)]}$$

This is an example of 'risk-neutral valuation'.

- Notice that $P$ does not appear in the formula, which is somewhat surprising. Shouldn't the option to buy something depend on the likelihood of the price going up?

# The Black-Scholes Formula

- The BS formula can be derived in several ways. We first derive it using a payoff replicating portfolio strategy. This is the way BS derived it. We then derive it using a more modern Risk-Neutral/EMM strategy.

- Assumptions
  1. Underlying stock price is geometric Brownian motion.
  2. Riskless interest rate is constant.
  3. No borrowing constraints or limits on short-selling.
  4. Stock does not pay dividends, or produce other cash flows.
  5. No arbitrage!

- Notation
  1. $S$ = Stock Price ($\frac{dS}{S} = \mu dt + \sigma dB$)
  2. $r$ = Riskless rate ($\frac{dR}{R} = r dt$).
  3. $K$ = Strike Price
  4. $T$ = Expiration date.
  5. $C(S,t)$ = Price of call option.

# Replicating Portfolio

- By Ito's lemma (subscripts denote partial derivatives),

$$\begin{aligned} dC &= C_s dS + \frac{1}{2}\sigma^2 S^2 C_{ss} dt + C_t dt \\ &= \left(\mu S C_s + \frac{1}{2}\sigma^2 S^2 C_{ss} + C_t\right) dt + \sigma S C_s dB \end{aligned}$$

- Form a portfolio, $P$, consisting of $x$ units of stock and $y$ units of bonds, so that $P = x \cdot S + y \cdot R$. Therefore,

$$\begin{aligned} dP &= x \cdot dS + y \cdot dR \\ &= x[\mu S dt + \sigma S dB] + yrRdt \\ &= (x\mu S + yrR)dt + x\sigma S dB \end{aligned}$$

- Pick $x$ and $y$ to replicate the option contract. Matching the $dB$ (diffusion) terms gives

$$x = C_s$$

  Now pick $y$ so that $P = C_s \cdot S + yR = C$. This gives,

$$y = R^{-1}(C - C_s \cdot S)$$

# The No Arbitrage PDE

- Plug $x$ and $y$ into the previous expression for $dP$

$$\begin{aligned} dP &= (x\mu S + yrR)dt + x\sigma S dB \\ &= [\mu S \cdot C_s + r(C - S \cdot C_s)]dt + \sigma S C_s \cdot dB \end{aligned}$$

- Note that the $dB$ terms are the same in $dP$ and $dC$. Matching the drift ($dt$) terms gives:

$$\mu S C_s + \frac{1}{2}\sigma^2 S^2 C_{ss} + C_t = \mu S C_s + r(C - S \cdot C_s)$$

Rearranging we get

$$\boxed{rC = C_t + rS \cdot C_s + \frac{1}{2}\sigma^2 S^2 \cdot C_{ss} \qquad \text{Black-Scholes PDE}}$$

- This must be solved subject to the *boundary conditions*:

$$\begin{aligned} C(S,T) &= \max\{S_T - K, 0\} \\ C(0,T) &= 0 \end{aligned}$$

# Conversion to Heat Equation

- In general, PDEs do not have analytical solutions. However, the BS PDE is *linear*, and it turns out that with appropriate transformations, it can be converted to the 'Heat Equation', which is one of the classical PDEs from physics, with a well known solution.
- To see this, define the following change of variables:

$$x = \log\left(\frac{S}{K}\right) \qquad \tau = \frac{1}{2}\sigma^2(T-t)$$

  and define $C(S,t) = K \cdot V(x,\tau)$.

- Using the chain rule

$$C_t = -\frac{1}{2}\sigma^2 K V_\tau \qquad C_s = \frac{K}{S}V_x \qquad C_{ss} = \frac{K}{S^2}\left(V_{xx} - V_x\right)$$

- This gives the following equation for $V(x,\tau)$

$$V_\tau = V_{xx} + (\kappa - 1)\cdot V_x - \kappa \cdot V$$

  where $\kappa = \frac{2r}{\sigma^2}$.

- Note that the boundary condition at $t = T$ becomes an initial condition at $\tau = 0$. Since $V = \frac{1}{K}C$, we get
$$V(x,0) = \max[e^x - 1, 0]$$
- Finally, let's change variables one more time, and define
$$V = e^{\alpha x + \beta \tau} U(x,\tau)$$
Using the chain rule, this gives the following equation for $U$
$$U_\tau = U_{xx} + [2\alpha + (\kappa - 1)]U_x + [\alpha^2 + (\kappa - 1)\alpha - \kappa - \beta]U$$
Note that if we choose $\alpha = -\frac{1}{2}(\kappa - 1)$ and $\beta = -\frac{1}{4}(\kappa + 1)^2$ we get
$$\boxed{U_\tau = U_{xx} \qquad \text{Heat Equation}}$$
with transformed initial condition
$$U(x,0) = \max\left[e^{\frac{1}{2}(\kappa+1)x} - e^{\frac{1}{2}(\kappa-1)x}, 0\right]$$

# Solving with Fourier Transforms

- The basic idea behind all transform methods is the following:
  1. Use the transform to convert a hard eq. into an easy eq.
  2. Solve the easy equation.
  3. Invert the transform to get solution of original eq.
- Given a function, $f(x)$, the Fourier transform and its inverse are

$$\begin{aligned}\mathcal{F}[f] &= F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\ \mathcal{F}^{-1}[F] &= f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega\end{aligned}$$

Fourier Transform Pair

- Note, the transform of derivatives get converted to multiplications

$$\begin{aligned}\mathcal{F}[f_x] &= i\omega\mathcal{F}[f] \\ \mathcal{F}[f_{xx}] &= -\omega^2\mathcal{F}[f] \\ \mathcal{F}[f_t] &= \frac{\partial}{\partial t}\mathcal{F}[f] \\ \mathcal{F}[f * g] &= \mathcal{F}[f]\mathcal{F}[g]\end{aligned}$$

where $(f * g)(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)g(s)ds$.

- This implies
$$\mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} = f * g \qquad \text{Convolution Property}$$

- Letting $\hat{U} = \mathcal{F}[U]$ we get
$$\frac{d\hat{U}}{d\tau} = -\omega^2 \hat{U}$$
Note that our PDE is now an ODE! The solution is:
$$\hat{U} = \hat{U}(0)e^{-\omega^2\tau}$$
where $\hat{U}(0) = \mathcal{F}[U(x,0)]$.

- Now we just need to invert, using the Convolution Property

$$\boxed{U(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(s,0)e^{\frac{-(x-s)^2}{4\tau}}ds \qquad \text{Soln. of Heat Eq.}}$$

- The function $e^{\frac{-(x-s)^2}{4\tau}}$ goes by various names. Physicists call it a ‘heat kernel’. Mathematicians call it a ‘Green’s function’. Economists call it an ‘impluse response function’.

- Finally, to get the BS formula, we just need to unwind the changes of variables

$$C(S,t) = S \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2) \qquad \text{Black-Scholes Formula}$$

where $N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-s^2/2} ds$ is the Normal CDF and

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

- Note, $(d_1, d_2) \to \infty$ as $t \to T$ if $S > K$, and $\to -\infty$ if $S < K$, which verifies the boundary condition

$$C(S,T) = \max\{S - K, 0\}$$

# An Alternative Replication Strategy

- BS actually derive their PDE using a different (but equivalent) portfolio strategy. Rather than form a portfolio of the stock and bond which replicates the option contract, they form a portfolio consisting of the stock and option which is riskless.
- Let $P = \theta_1 C(S,t) + \theta_2 S$ be the value of portfolio consisting of $\theta_1$ units of the option and $\theta_2$ units of the stock.
- From Ito's lemma,

$$dP = \theta_1 [C_t + C_s dS + \frac{1}{2}\sigma^2 S^2 C_{ss}]dt + \theta_2 dS$$

- Suppose we set $\theta_1 = 1$ and $\theta_2 = -C_s$. Note that the terms involving $dS$ cancel, and we get (note also that we could instead set $\theta_1 = -1$ and $\theta_2 = C_s$)

$$dP = (C_t + \frac{1}{2}\sigma^2 S^2 C_{ss})dt$$

This is non-random, and so is riskless. Hence, to avoid arbitrage we must have

$$dP = rPdt = r(C - S \cdot C_s)dt = (C_t + \frac{1}{2}\sigma^2 S^2 C_{ss})dt$$

This then implies

$$rC = C_t + rS \cdot C_s + \frac{1}{2}\sigma^2 S^2 C_{ss}$$

which is the same PDE as before!

# DERIVING THE BS FORMULA USING EMMS

- Clearly, deriving the BS formula using portfolio replication and PDEs involves some lengthy mathematical manipulations.

- A simpler, more direct, and more modern derivation is based on EMMs. Black-Scholes didn't use this method because it hadn't been invented yet! In fact, their work was the inspiration for it.

- The key idea is that with a single shock, the market is effectively completed by dynamic trading in the underlying stock and a riskless bond. (Note, the stock price need not follow the geometric Brownian motion posited by BS, although if there are additional stochastic state variables affecting its drift or diffusion, then we must use additional assets to price the option using no arbitrage). The martingale conditions associated with the stock and bond determine a unique SDF process for the price of any derivative written on the underlying stock. (Note - we need not assume the entire market is complete. Any nonuniqueness in the SDF will be orthogonal to the underlying asset, and so be irrelevant in determining its price).

- Following the usual no arbitrage logic, we know

$$C(S,t) = E_t\left[\left(\frac{M_T}{M_t}\right)\max[S_T - K, 0]\right]$$

where $\frac{dM}{M} = -rdt - \kappa dB$. If the stock follows a geometric Brownian motion, then $\kappa = (\mu - r)/\sigma$.

- Without loss of generality, suppose $t = 0$, and the BS assumptions apply (ie, constant riskless rate, and geometric Brownian motion). Also let $A =$ event that $S_T > K$. Remembering that $M_0 = 1$, we have

$$\begin{aligned} C(S, 0) &= E[M_T S_T \cdot 1_A] - KE[M_T \cdot 1_A] \\ &= e^{-rT} E^Q[S_T \cdot 1_A] - e^{-rT} K E^Q[1_A] \end{aligned}$$

where $E^Q$ denotes expectations taken with respect to the EMM, $\xi = MR$.

- With respect to the risk-neutral measure, $Q$, the stock is given by

$$\log S_T = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma \hat{B}_T = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\varepsilon$$

where $\varepsilon \sim N(0, 1)$ under $Q$. Also note that $A$ is equivalent to the event

$$\varepsilon > -d_2 = -\left[\frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

Therefore,

$$\begin{aligned} E^Q[S_T \cdot 1_A] &= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}\varepsilon} e^{-\varepsilon^2/2} d\varepsilon \\ &= e^{rT} S_0 \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(\varepsilon - \sigma\sqrt{T})^2/2} d\varepsilon \\ &= e^{rT} S_0 \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = e^{rT} S_0 [1 - N(-d_1)] = e^{rT} S_0 N(d_1) \end{aligned}$$

where $y = \varepsilon - \sigma\sqrt{T}$ and $d_1 = d_2 + \sigma\sqrt{T}$.

- Following the same steps, one can easily see that $E^Q[1_A] = N(d_2)$. Hence, plugging these results into the risk-neutral evaluation of the option we get back the BS formula

$$C(S,0) = e^{-rT}E^Q[S_T \cdot 1_A] - e^{-rT}KE^Q[1_A] = S_0 N(d_1) - e^{-rT}KN(d_2)$$

- The derivation can be further simplified if we use two EMMs. In addition to the traditional EMM, $\xi_T = M_T \cdot R_T$, use the *other* martingale condition to define a new EMM, $\xi_T^s = M_T S_T / S_0$. Note that with respect to this EMM, the stock price follows (see if you can verify this):

$$\frac{dS}{S} = (r + \sigma^2)dt + \sigma d\tilde{B}$$

- Denoting expectations with respect to this measure by $E^{Q^s}$ we get the simple formula

$$\boxed{C(S,0) = S_0 E^{Q^s}[1_A] - e^{-rT}KE^Q[1_A] \qquad \text{BS using EMMs}}$$

- One can readily verify that this once again delivers the BS formula.

# A Feynman-Kac Solution

- In prob set 2 you learned that if $x$ follows the diffusion $dx = \mu(x,t)dt + \sigma(x,t)dB$ then $g(x,t) = E[G(x_T)|x_t = x]$ satisfies the PDE

$$0 = g_t(x,t) + \mu(x,t)g_x(x,t) + \frac{1}{2}\sigma^2(x,t)g_{xx}(x,t)$$

  subject to the boundary condition $g(x,T) = G(x_T)$.

- A slight generalization incorporates discounting, $g(x,t) = E\left[e^{-\int_t^T r_s ds}G(x_T)|x_t = x\right]$. Following the same steps as in prob set 2, one can easily verify that $g$ obeys the modified PDE

$$r_t g(x,t) = g_t(x,t) + \mu(x,t)g_x(x,t) + \frac{1}{2}\sigma^2(x,t)g_{xx}(x,t)$$

- Observe that the BS PDE is just a special case of this, with $x = S$, $G = \max[S_T - K, 0]$, $r$ constant, $\mu(S,t) = rS$, and $\sigma(S,t) = \sigma S$.

- <u>Punchline</u>: All (path independent) derivatives on $S$ satisfy the same PDE! All that differs is the boundary condition. BS picked a special case that delivers an analytical solution, but the PDE itself is much more general.

- The key insight from BS is that no arbitrage implies the drift of the underlying is equal to the riskless rate.

# One Last Derivation of Black-Scholes

- As noted in Prob Set 2, in the question about the Feynman-Kac formula, there is a close connection between 2nd-order PDEs and expected values of diffusion processes, which follows directly from Ito's lemma. When deriving the BS formula, we solved the PDE by brute force, using analytical methods invented in the early 1800s.

- These days, with fast computers, it is easy to compute the expected value directly, via monte carlo simulation. Instead of solving PDEs, we can compute no arbitrage prices by numerically simulating expected values. Expected values are just *averages*, and we can obtain good approximations by generating many 'random' realizations of $S$, and then just computing the average discounted payoff. The only subtlety is which discount factor to use.

- By now, we know we have two choices. (1) We can simulate paths of $S$ using the objective prob measure, and then discount the terminal payoff using the SDF, $M_T$, or (2) We can simulate paths of $S$ using the risk-neutral measure, and then discount using the riskless rate.

- The beauty of the risk-neutral approach is that we do not even need to know the mean return on the stock! We just need to know the riskless rate. Consequently, this is the approach used in practice.

- Here is an outline, for the case of geometric Brownian motion: First, divide the interval $[0, T]$ into steps of length $\Delta t$. Then, for $i = 1$ to SimNum,
  1. Generate $T/\Delta t$ draws from a $N(0, \Delta t)$ distribution.
  2. Compute a sequence of stock prices using the risk-neutral measure:

$$S(t_n + \Delta t) = S(t_n) + rS(t_n)\Delta t + \sigma S(t_n)\varepsilon(t_n) \qquad n = 1, 2, \cdots T/\Delta t$$

  3. Compute $f(S(T))$, where $f(\cdot)$ is the boundary condition. For a call it is $f(S(T)) = \max[S(T) - K, 0]$.
  4. At the end of the loop compute

$$\hat{C} = e^{-rT}\text{average}[f(S(T))]$$

- This monte carlo approach easily handles general $S$ processes. It does not need to be geometric Brownian motion.
- It easily handles complicated (even path dependent!) boundary conditions.
- However, it does not easily handle American-style options, which may be exercised early. Nor is it well suited to 'real-time' decision-making and high-frequency trading.

# EXOTIC OPTIONS

Exchange traded options are pretty standardized (calls, puts, and a few variants). However, in the OTC market, the sky's the limit, and there are a bewildering variety of exotic options:

1. Lookback Options: The strike price is determined by the *minimum* value of the stock (or max for a put).
2. Asian Options: The payoff depends on the *average* value of of the stock. Ex. 1: $C_T = \max[S_T - S_{avg}, 0]$. Ex. 2: $C_T = \max[S_{avg} - K, 0]$.
3. Knockout Options: The options expires immediately if the asset price hits a barrier. A rebate is sometimes paid.
4. Forward Start Options: The option is purchased at one date, but does not become active until a later date.
5. Chooser Options: After a specified time, the holder can choose whether it's a call or a put.
6. Compound Options: An option on an option.

# Options Aren't Riskless!

- Although priced using no arbitrage/risk-neutral methods, options are most certainly not riskless assets. (Of course, a dynamically adjusted portfolio of an option and the underlying asset can be made riskless).

- In fact, buying a call is akin to buying the underlying stock on margin. It is a leveraged bet. Remember that the number of units of the riskless asset in the replicating portfolio was, $y = R^{-1}(C - S \cdot C_s)$. From BS, we now know that $C_s = N(d_1)$. This is the number of shares held in the underlying asset. Therefore,

$$S \cdot C_s = S \cdot N(d_1) > C = S \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2)$$

  Thus, to replicate the call, you borrow and invest the proceeds in the stock. (Since $C_{ss} > 0$, as the price rises you borrow even more and buy more shares!)

- We can calculate the mean return from holding the call as follows

$$E\left[\frac{1}{dt}\left(\frac{dC}{C}\right)\right] = \frac{1}{C}\left[C_t + \mu S \cdot C_s + \frac{1}{2}\sigma^2 S^2 \cdot C_{ss}\right] \equiv \mu_c$$

  Using the BS PDE to sub out for $C_t$, we get

$$\mu_c = \frac{1}{C}\left[(\mu - r)SC_s + rC\right]$$

- Next, we know the diffusion coefficient of the call is

$$\sigma_c = \frac{1}{C}\left[\sigma S \cdot C_s\right]$$

Thus, the Sharpe ratio of the call is

$$\frac{\mu_c - r}{\sigma_c} = \frac{\frac{1}{C}[(\mu - r)SC_s + rC] - r}{\frac{1}{C}\sigma S C_s} = \frac{\mu - r}{\sigma}$$

As expected, given the completeness of the market, the Sharpe ratio of the call is the same as the Sharpe ratio of the underlying asset. Let $\kappa =$ this common Sharpe ratio. Then we can write

$$\mu_c = r + \sigma_c \kappa$$

But we know from above that $\sigma_c > \sigma$, since $SC_s > C$. Hence, we know $\mu_c > \mu$, i.e., the expected return on the call exceeds the expected return on the stock. This is perhaps why some naive traders like options!

- Interestingly, everything is reversed for puts. A put is akin to short position in the stock, with proceeds invested in the riskless asset. Its risk premium is lower, $\mu_p < \mu$. Intuitively, buying a put is like buying insurance. The lower rate of return can be interpreted as an insurance premium.

# Extensions & Generalizations

In a sense, the BS formula is relatively robust, since it is based on simple no arbitrage logic. However, to derive analytic formulas, one has to make assumptions. Many finance professors have devoted their careers to relaxing these assumptions. Among the more important are:

1. Time-Varying Interest Rate: Punchline: If you are pricing a 6-month option, use the 6-month T-bill rate. If you are pricing a 1-year option, use the 1-year T-bill rate, etc.

2. Time-Varying Volatility: If volatility is a *deterministic* function of $t$ and $S$, $\sigma(S, t)$, no new issues arise (although the BS formula will need to be modified unless $\sigma(S, t) = \sigma S$). *Stochastic* volatility causes significant changes. If it's uncorrelated with the asset price, BS remains valid with an appropriate averaging procedure. Correlated stochastic volatility introduces new hedging requirements, and in principle requires additional assets for replication. If we make an *assumption* about the price of volatility risk, we can simply add a new parameter to the SDF/EMM, and proceed as before.

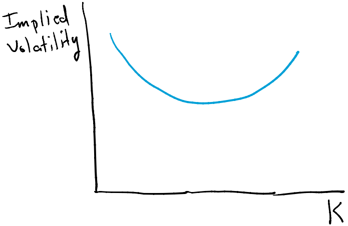
3. Dividends: In general, if a stock pays dividends you can no longer derive an analytic formula. However, in some special cases you can. For example, if dividends are a constant fraction of the stock price (constant dividend yield), then only a simple adjustment in the BS formula is required. This is a decent approximation for index options, but questionable for individual stocks. Of course, if you *know* the stock will not pay dividends during the life of the option, then the standard BS formula applies.

4. Transaction Costs: All hell breaks loose. (You can no longer maintain a riskless portfolio). In practice, the best you can do is derive bounds.

# Empirical Evaluation

- In principle, the BS formula should hold exactly, since it's based on no arbitrage. Although it works well, there are discrepancies.
- One obvious suspect is the assumptions about transaction costs. However, the fact that it often does work, suggests that this might not be the main issue.
- Note that all the BS parameters, $(S, K, T, r, \sigma)$, are observable, except one, namely, $\sigma$.
- We can always make the BS formula fit perfectly if we pick $\sigma$ appropriately. For a given option, this value is called the implied volatility.
- However, there are many options written on the same stock. They should all give the same implied volatility. In practice, they often differ. There is an intriguing pattern that is often generated, called the Volatility Smile.

# Volatility Smile




The volatility smile suggests that deep in-the-money and deep out-of-the-money options are ‘over priced’. The smile is most prevalent in currency options.

Equity index implied volatilities are often higher for low strikes than for high strikes, so the smile becomes a ‘smirk’. The smirk implies inordinately high prices for deep out-of-the-money puts. Some argue that this reflects omitted left-tail/crash risk from the BS formula (which presumes a symmetric distribution of returns).

# Black-Scholes and Corporate Finance

- Finally, options pricing is applicable to the valuation of corporate debt and equity.
- BS observed that shareholders effectively own a call option on the assets of a firm. When a firm's debt matures, the shareholders can pay off the debt (i.e., exercise the call) and become sole claimants to the firm's assets. However, if the firm is worth less than the outstanding debt, the call is out-of-the-money and shareholders don't exercise.
- From Put/Call Parity, shareholders can equivalently be viewed as owning a put rather than a call. Shareholders own the firm, are short cash (the face value of the debt), and have an option to put the firm to the bondholders by declaring bankruptcy.
- Hence, if a firm's asset values are lognormal, the BS formula can be used to value corporate debt and equity. In practice, the implicit options embedded in corporate debt and equity are American rather than European, so extensions to BS which allow potential early exercise become important.