NEW YORK UNIVERSITY
Department of Economics

Econ-GA 2021 — Prof. Kasa
Financial Economics I — Fall 2018

PROBLEM SET 1
(Solutions)

Each of the following questions is worth 10 points.

1. Download the dataset ProbSet1.xls from the class website. The final 2 columns contain (nominal) monthly returns on the 'market portfolio' (i.e., value-weighted portfolio of stocks listed on the NYSE, AMEX, and NASDAQ stock exchanges), and the 30-day Treasury Bill rate. (You can ignore the other 4 return series). The data run from July 1926 to June 2016.

   (a) Plot the market return and the T-Bill rate.

   (b) What were the annualized mean stock market and T-Bill returns? (Hint: Multiply the mean monthly returns by 12). What were their standard deviations? (Hint: Multiply the standard deviations of monthly returns by $\sqrt{12}$).

   The mean annualized return on the market was 11.12%, *(remember, this is nominal), with a standard deviation of* 18.6%. *The annualized mean T-Bill return was* 3.35%, *with a standard deviation of only* 0.88%.

   (c) What was the mean equity premium? Is their any evidence that it has changed over time? Split the sample in half, and compute the mean equity premium in each half. Any difference?

   Given the above results, the mean equity premium was $11.12 - 3.35 = 7.77\%$. *There is some evidence that it has declined over time. In the first half of the sample (until 1971) it was* 9.23%, *whereas in the latter half of the sample it was only* 6.30%.

   (d) What was the mean Sharpe ratio? Has it changed over time?

   There is less evidence that the Sharpe ratio has changed. The full sample Sharpe ratio was .417, *while the first and second half Sharpe ratios were* .437 *and* .401. *This is because the volatility of the market has declined at the same time as the equity premium has.*

2. Assume there are two possible states of the world, s_1 and s_2. There are two assets: (1) a risk-free asset with an initial price of one, that pays R_f in each state, and (2) a risky asset with initial price one that pays R_d in state s_1 and R_u in state s_2. Assume without loss of generality that $R_u > R_d$.

   (a) What must be the relationship among (R_u, R_d, R_f) for there to be no arbitrage opportunities.

   The mapping between prices and payoffs is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_f & R_f \\ R_u & R_d \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

   Without loss of generality, assume $R_u > R_d$, *so that state 1 is the 'good' state. Solving for* (q_1, q_2) *gives*

$$q_1 = \frac{R_f - R_d}{R_f(R_u - R_d)} \qquad q_2 = \frac{R_u - R_f}{R_f(R_u - R_d)}$$

   No arbitrage requires $q_i > 0$. *Hence, the no arbitrage condition is* $R_u > R_f > R_d$.

(b) Assuming no arbitrage, compute the unique vector of state prices. Also compute the unique risk-neutral probabilities of states s_1 and s_2.

The state prices were already calculated when solving part (a). The risk-neutral probabilities just normalize the state prices to sum to unity. Note that $q_1 + q_2 = R_f^{-1}$. *Hence, the price of a sure claim is the reciprocal of the risk-free rate (as always). Normalizing the state prices by dividing by their sum gives the following risk-neutral probabilities,* q_i^*

$$q_1^* = \frac{R_f - R_d}{R_u - R_d} \qquad q_2^* = \frac{R_u - R_f}{R_u - R_d}$$

Note that $q_1^* + q_2^* = 1$.

(c) Now consider an option contract on the risky asset, which pays $\max[x - K, 0]$ for some constant K, where $x \in \{R_u, R_d\}$ is the unknown future price/payoff of the risky asset. Compute the no arbitrage price of this option.

The no arbitrage price of the call is just the expected value of its payoff, discounted at the risk-free rate, and where expectations are computed using the risk-neutral probabilities. This gives

$$P_{call} = \frac{1}{R_f}\left[\left(\frac{R_f - R_d}{R_u - R_d}\right) \cdot \max[R_u - K, 0] + \left(\frac{R_u - R_f}{R_u - R_d}\right) \cdot \max[R_d - K, 0]\right]$$

Later we will generalize this procedure to continuous-time stochastic processes.

3. Suppose you have a single return, R. By projecting onto the span of assets, we know $m_p = R/E(R^2)$ is one possible stochastic discount factor. (See, e.g., p. 64 in Back). What about R^{-1}? Clearly, $E(R^{-1}R) = 1$, so doesn't this violate the claim that m_p is unique? Do these stochastic discount factors rule out arbitrage?

R^{-1} *is indeed a valid discount factor in this case. However, it is not a linear combination of the assets. The theorem only says that* m_p *is the unique linear combination of the underlying assets. Neither one of these discount factors rules out arbitrage (in general), since they can both be negative.*

4. Suppose a stock is currently worth \$20, and it is known that in 3 months it will be worth either \$22 or \$18. Consider on option on the stock with a strike price of \$21. This option will either be worth \$1 (if the stock price increases) or worth nothing (if the stock price decreases). This question asks you to use no arbitrage reasoning to value this option contract. (Later we shall generalize this argument to much more complicated settings).

(a) Consider a portfolio consisting of a long position of Δ shares of the stock and a short position of one call option. Find a value of Δ that makes this a riskless portfolio (i.e., its payoff is the same, no matter what the future stock price turns out to be).

If the stock moves up, the value of the shares is 22Δ *and the value of option is* 1, *so the value of the portfolio is* $22\Delta - 1$. *If the stock goes down, the value of the shares is* 18Δ *and the value of the option is* 0, *so the value of the portfolio is* 18Δ. *The portfolio is riskless if it has the same value no matter what the stock price does. This requires*

$$22\Delta - 1 = 18\Delta$$

which implies $\Delta = 0.25$.

(b) Given this value of Δ, what will be the future value of the portfolio? Assuming the (annual) risk-free interest rate is 12%, what is the present value of this portfolio?

The future value of the portfolio will be $18(.25) = 4.5$. *Since the portfolio is riskless, it must earn the riskfree rate. Hence, its present value is*

$$4.5\left(e^{-0.12\times 3/12}\right) = 4.367$$

(c) What must therefore be the current no arbitrage price of the option? If the price deviated from this value, explain how you could make riskless profits.

Remember that the current price of the stock is 20. *If we let* c *denote the current price of the call option, then by no arbitrage we have*

$$4.367 = 20(0.25) - c \qquad \Rightarrow \qquad c = 0.633$$

(d) Why didn't we need to know the probability that the stock price would increase? Wouldn't this influence your valuation of a call option?

There are a couple of ways to think about this. Intuitively, the option is a redundant security here. Its payoff can be perfectly replicated by holding the stock and the bond. Hence, all that matters for its valuation are the risk-neutral probabilities. More formally, and more subtlely, the drift of the stock does *matter, but in two exactly opposing ways. A higher drift increasing the expected future value of the stock, thus raising the expected payoff, but it also increases the price of risk, which increases the discount rate. In the Black-Scholes world, these two exactly offset each other.*

5. Suppose there are two states of the world, s_1 and s_2. Also suppose there are two assets: (1) A risky asset that pays 1 unit in s_1 and 3 units in s_2, and (2) A riskless asset that pays 1 unit in both states. Assume the riskless asset is in zero net supply. There are two agents: (1) A risk neutral agent with utility function $U = E(c)$, and (2) A risk averse agent with utility function $U = E\sqrt{c}$. Both agents assign equal probabilities to s_1 and s_2, and each is endowed with half the shares of the risky asset. Solve for the competitive equilibrium (relative) price of the risky asset, and compute the equilibrium allocation of the two securities. Explain your results intuitively.

Let θ_{1i} *be agent-*i*'s holdings of the risk-free asset, and* θ_{2i} *be agent-*i*'s holdings of the risky asset. Hence, the market-clearing conditions are:* $\theta_{11} + \theta_{12} = 0$ *and* $\theta_{21} + \theta_{22} = 1$. *Let* P *denote the relative price of the risky asset. The two agents' problems are then given by*

$$Agent1: \max_{\theta_{11},\theta_{21}} \left\{\frac{1}{2}(\theta_{11} + \theta_{21}\cdot 1) + \frac{1}{2}(\theta_{11} + \theta_{21}\cdot 3) + \lambda_1\left(\frac{1}{2}P - \theta_{11} - P\theta_{21}\right)\right\}$$

$$Agent2: \max_{\theta_{12},\theta_{22}} \left\{\frac{1}{2}(\theta_{12} + \theta_{22}\cdot 1)^{1/2} + \frac{1}{2}(\theta_{12} + \theta_{22}\cdot 3)^{1/2} + \lambda_2\left(\frac{1}{2}P - \theta_{12} - P\theta_{22}\right)\right\}$$

where λ_i *is the Lagrange multiplier on agent-*i*'s budget constraint. The 4 FOCs along with the 2 market-clearing conditions and 2 budget constraints give 8 equations in the 7 unknowns* $(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}, \lambda_1, \lambda_2, P)$. *As usual, we can drop one equation due to Walras Law. Note that the FOC w.r.t,* θ_{11} *implies* $1 - \lambda_1 = 0$, *while the FOC w.r.t.* θ_{21} *implies* $2 - \lambda_1 P = 0$. *Hence, we have* $P = 2$. *The presence of the risk neutral agent means that the price of the risky asset just equals its expected payoff. Solving for the portfolio shares we find*

$$\begin{aligned} \theta_{11} &= -1 \\ \theta_{21} &= 1 \\ \theta_{12} &= 1 \\ \theta_{22} &= 0 \end{aligned}$$

Thus, the risk-neutral agent ends up holding all of the risky asset. The risk averse agent sells his risky share in exchange for a risk-free bond. Effectively, the risk-neutral agent fully insures the risk averse agent.

6. This question explores the conditions that enable you to use observed asset prices to infer the (subjective) beliefs of market participants. Suppose there is a representative agent with time separable utility $U(\cdot)$ over consumption and time discount factor, δ. The investor's Euler equations price observed assets.

   (a) First consider a stationary economy with no growth. Suppose there are two consumption states, C_H and C_L. Transitions between the two states follow a Markov process with transition probabilities f_{ij} for $i, j \in \{H, L\}$. Denote the four Arrow-Debreu security prices by p_{ij}. That is, p_{ij} is the price of a claim to one unit of consumption in state j tomorrow given state i today. Write down the four optimality (Euler) equations of the agent. Use them to solve for the ratio of marginal utilities, the time discount factor, and the transition probabilities as functions of the state prices, p_{ij}.
   The agent's FOCs are

$$\begin{aligned} P_{HL}U'(C_H) &= \delta f_{HL}U'(C_L) \\ P_{HH}U'(C_H) &= \delta f_{HH}U'(C_H) \\ P_{LH}U'(C_L) &= \delta f_{LH}U'(C_H) \\ P_{LL}U'(C_L) &= \delta f_{LL}U'(C_L) \end{aligned}$$

   Letting $x = U'(C_L)/U'(C_H)$ and using the facts that $f_{HL} = 1 - f_{HH}$ and $f_{LH} = 1 - f_{LL}$, we can write these as

$$\begin{aligned} P_{HL} &= \delta(1 - f_{HH})x & (1) \\ P_{HH} &= \delta f_{HH}(1/x) & (2) \\ P_{LH} &= \delta(1 - f_{LL})(1/x) & (3) \\ P_{LL} &= \delta f_{LL}x & (4) \end{aligned}$$

   Dividing (1) by (4) and (3) by (2) we obtain

$$\begin{aligned} \frac{P_{HL}}{P_{LL}} &= \frac{1 - f_{HH}}{f_{LL}} & (5) \\ \frac{P_{LH}}{P_{HH}} &= \frac{1 - f_{LL}}{f_{HH}} & (6) \end{aligned}$$

   These are easily solved for f_{HH} and f_{LL}

$$f_{HH} = \frac{1 - P_{HL}/P_{LL}}{1 - (P_{HL}/P_{LL})(P_{LH}/P_{HH})} \qquad f_{LL} = \frac{1 - P_{LH}/P_{HH}}{1 - (P_{HL}/P_{LL})(P_{LH}/P_{HH})}$$

   This is interesting and important. Under these conditions we can use observed asset prices to separate agents' subjective beliefs from their preferences.

   (b) Now introduce consumption growth. Assume that given the current level of consumption, consumption grows at either rate g_H or g_L. Consumption growth follows a Markov process with transition probabilities f_{ij} for $i, j \in \{H, L\}$. Finally, assume the representative agent has time-additive CRRA preferences, with coefficient of relative risk aversion, γ

(i) Show that the four Arrow-Debreu prices are independent of the current level of consumption. Solve for γ, δ, and f_{ij} as functions of the state prices and growth rates, g_H and g_L.

This is really the same question as before. In eqs. (1)-(4), we just need to set $x = (1+g_L)^{-\gamma}$ *and* $1/x = (1+g_H)^{-\gamma}$. *Note that we get the same solutions for* f_{ij}.

(ii) Now suppose consumption growth is i.i.d (i.e, $p_{LH} = p_{HH}$ and $p_{LL} = p_{HL}$). Show that recovery of beliefs from asset prices breaks down.

Notice from eqs. (5)-(6) that if consumption growth is i.i.d., then the eq. system becomes degenerate (the l.h.s. of both equations equals 1). We can no longer solve uniquely for the f_{ij}.