NEW YORK UNIVERSITY
Department of Economics

Econ-GA 2021 | Prof. Kasa
Financial Economics I | Fall 2018

PROBLEM SET 2
(Due October 15)

Each of the following questions is worth 10 points.

1. This question asks you to numerically validate the total and quadratic variation properties of Brownian motion. Simulate a continuous-time Brownian on the unit interval $[0, 1]$ by dividing the interval into N equal subintervals, $\Delta t = 1/N$. Discretize the path of Brownian motion as follows: $B_{t_i} = B_{t_{i-1}} + \varepsilon_i \sqrt{\Delta t}$, with $\varepsilon_i \sim N(0, 1)$ and $B_0 = 0$. Perform the simulation for $N = 20, 100, 1000$.

   (a) Plot the paths of B_{t_i}.

   (b) For each path, approximate the quadratic variation by calculating the sum of $(\Delta B_{t_i})^2$, and confirm that for large N it converges to 1.

   (c) Calculate the total variation by calculating the sum of $|\Delta B_{t_i}|$. Confirm that it increases with N.

2. **Ito's Lemma Practice**.

   (a) Use Ito's lemma to verify that $X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$ is the solution to SDE $\frac{dX}{X} = \mu dt + \sigma dB_t$, where μ and σ are constants.

   (b) Given the SDE $\frac{dX}{X} = \mu dt + \sigma dB_t$, use Ito's lemmas to calculate dY for $Y = X^\alpha$.

   (c) Let B_t be a Brownian motion, and assume $X_t = B_t^2$. Use Ito's lemma to find a SDE for X_t.

   (d) Let B_t be a Brownian motion, and assume $X_t = 2 + t + e^{B_t}$. Use Ito's lemma to find a SDE for X_t.

   (e) Solve the Ornstein-Uhlenbeck SDE, $dX_t = -\mu X_t dt + \sigma dB_t$. Use the solution to compute $E[X_t]$ and $\text{var}[X_t] = E[(X_t - E[X_t])^2]$. (Hint: Multiply both sides by the integrating factor $e^{\mu t}$ and use Ito's lemma to compare with $d(e^{\mu t} X_t)$).

3. **Martingale Scaling**. Assume X_t follows the Ito process, $dX_t = \mu_t dt + dB_t$. Define the scaled process $Y_t = M_t X_t$, where

$$M_t = \exp\left(-\int_0^t \mu_s dB_s - \frac{1}{2} \int_0^t \mu_s^2 ds \right)$$

   Use Ito's lemma to compute dY and show that it is a (local) martingale. State the conditions that must be satisfied for Y_t to be well defined.

4. **Betting on a Brownian motion**. In class we discussed why ruling out arbitrage in continuous-time required restrictions on portfolio/betting strategies. This question considers a variant of the doubling strategy we studied. Let B_t be a Brownian motion, and define the stopping time $\tau = \inf\{t \geq 0 : B_t = 1\}$. We showed in class that $P[\tau < \infty] = 1$. Given this, consider the simple strategy of betting a *fixed* amount until $B_t = 1$. This also seems like a sure bet. Can you see any difference from the doubling strategy? Explain your reasoning both intuitively, and more formally using the distinction between martingales and local martingales. (Hint: Is it still the case here that $E_0[B_{\tau \wedge t}] = 1$?).

5. **The Feynman-Kac Formula**. There is a close relationship between 2nd-order partial differential equations and conditional expectations of diffusion processes. This relationship is revealed by the Feynman-Kac formula. The FK formula is widely used in financial economics.

   Consider the scalar diffusion process

   $$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

   Suppose we want to compute the expected terminal payoff $G(X_T)$ at some future date T, given the current value of the process (e.g., we want to value an option)

   $$g(x, t) = E[G(X_T)|X_t = x]$$

   Prove that g can be obtained by solving the following 2nd-order (linear) PDE

   $$0 = g_t(x, t) + g_x(x, t)\mu(x, t) + \frac{1}{2}g_{xx}(x, t)\sigma^2(x, t)$$

   with boundary condition, $g(x, T) = G(x)$.

   (Hint: Use the law of iterated expectations to write

   $$\begin{aligned} g(x, t) &= E[E[G(X_T)|X_{t+dt}]|X_t = x] \\ &= E[g(X_{t+dt}, t + dt)|X_t = x] \end{aligned}$$

   and then use Ito's lemma to write $g(X_{t+dt}, t + dt)$ in terms of $g(x, t)$ and its derivatives).

6. **Stochastic Growth with CARA utility**. In class we solved a stochastic growth model with CRRA utility. This problem asks you to solve a stochastic growth model with CARA preferences.

   Consider an agent who wants to solve the following problem

   $$\max_c E\int_0^\infty e^{-\rho t}u(c)dt \qquad \text{where} \qquad u(c) = \frac{-1}{\gamma}e^{-\gamma c}$$

   where the parameter γ is the coefficient of absolute risk aversion. The capital stock, k, evolves according to the following stochastic differential equation

   $$dk = (\mu k - c) \cdot dt + \sigma dB$$

   where dB is an increment to a Brownian motion process, and μ and σ are constant parameters.

   (a) Write down the agent's (stationary) HJB equation.

   (b) Use a guess-and-verify strategy to solve the HJB equation. (Hint: Try the guess $V(k) = -\gamma^{-1}e^{Ak+B}$ where A and B are undetermined coefficients).

   (c) Given your answer to part (b), write down the agent's optimal consumption/savings policy. Interpret your answer in terms of intertemporal substitution and precautionary saving.