NEW YORK UNIVERSITY
Department of Economics

Econ-GA 2021 Prof. Kasa
Financial Economics I Fall 2018

PROBLEM SET 2
(Solutions)

Each of the following questions is worth 10 points.

1. This question asks you to numerically validate the total and quadratic variation properties of Brownian motion. Simulate a continuous-time Brownian on the unit interval $[0, 1]$ by dividing the interval into N equal subintervals, $\Delta t = 1/N$. Discretize the path of Brownian motion as follows: $B_{t_i} = B_{t_{i-1}} + \varepsilon_i \sqrt{\Delta t}$, with $\varepsilon_i \sim N(0, 1)$ and $B_0 = 0$. Perform the simulation for $N = 20, 100, 1000$.

   (a) Plot the paths of B_{t_i}.

   (b) For each path, approximate the quadratic variation by calculating the sum of $(\Delta B_{t_i})^2$, and confirm that for large N it converges to 1.

   (c) Calculate the total variation by calculating the sum of $|\Delta B_{t_i}|$. Confirm that it increases with N.

2. **Ito's Lemma Practice**.

   (a) Use Ito's lemma to verify that $X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$ is the solution to SDE $\frac{dX}{X} = \mu dt + \sigma dB_t$, where μ and σ are constants.

   Defining $X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma B_t} = F(t, B)$ *we have*

$$F_t = \left(\mu - \frac{1}{2}\sigma^2\right) X \qquad F_B = \sigma X \qquad F_{BB} = \sigma^2 X$$

   From Ito's lemma, $dX = F_t dt + F_B dB + \frac{1}{2} F_{BB}(dB)^2$. *Plugging in and simplifying gives* $\frac{dX}{X} = \mu dt + \sigma dB_t$

   (b) Given the SDE $\frac{dX}{X} = \mu dt + \sigma dB_t$, use Ito's lemmas to calculate dY for $Y = X^\alpha$.

   Defining $Y = X^\alpha = F(X)$ *and applying Ito's lemma,*

$$\begin{aligned} dY &= F_X dX + \frac{1}{2} F_{XX}(dX)^2 \\ &= \alpha X^{\alpha-1}[\mu X dt + \sigma X dB] + \frac{1}{2}\alpha(\alpha - 1) X^{\alpha-2} \sigma^2 X^2 dt \\ &= \alpha X^\alpha[\mu dt + \sigma dB] + \frac{1}{2}\alpha(\alpha - 1)\sigma^2 X^\alpha dt \end{aligned}$$

   Therefore, we get

$$\frac{dY}{Y} = \left[\alpha\mu + \frac{1}{2}\alpha(\alpha - 1)\sigma^2\right] dt + \alpha\sigma dB$$

   (c) Let B_t be a Brownian motion, and assume $X_t = B_t^2$. Use Ito's lemma to find a SDE for X_t.

   Defining $X = B^2 = F(B)$ *and applying Ito's lemma we get*

$$dX = dt + 2B dB$$

(d) Let B_t be a Brownian motion, and assume $X_t = 2 + t + e^{B_t}$. Use Ito's lemma to find a SDE for X_t.

Defining $X = 2 + t + e^B = F(t, B)$ and applying Ito's lemma we get

$$dX = \left(1 + \frac{1}{2}e^B\right) dt + e^B dB$$

(e) Solve the Ornstein-Uhlenbeck SDE, $dX_t = -\mu X_t dt + \sigma dB_t$. Use the solution to compute $E[X_t]$ and $\text{var}[X_t] = E[(X_t - E[X_t])^2]$. (Hint: Multiply both sides by the integrating factor $e^{\mu t}$ and use Ito's lemma to compare with $d(e^{\mu t} X_t)$).

First, note that $d(e^{\mu t} X) = \mu e^{\mu t} X dt + e^{\mu t} dX$. Next, if we multiply both sides of the Ornstein-Uhlenbeck equation by $e^{\mu t}$ we get, $e^{\mu t} dX = -\mu X e^{\mu t} dt + e^{\mu t} \sigma dB$. Subbing this into the previous equation gives

$$d\left(e^{\mu t} X\right) = e^{\mu t} \sigma dB$$

Integrating gives

$$e^{\mu t} X_t - X_0 = \sigma \int_0^t e^{\mu s} dB_s$$

Rearranging gives the solution of the OU equation,

$$X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s$$

Therefore, $E(X_t) = X_0 e^{-\mu t}$ and

$$\text{var}(X_t) = E[(X - E(X))^2] = E\left[\sigma \int_0^t e^{-\mu(t-s)} dB_s\right]^2 = \sigma^2 \int_0^t e^{-2\mu(t-s)} ds = \frac{\sigma^2}{2\mu}\left[1 - e^{-2\mu t}\right]$$

3. **Martingale Scaling**. Assume X_t follows the Ito process, $dX_t = \mu_t dt + dB_t$. Define the scaled process $Y_t = M_t X_t$, where

$$M_t = \exp\left(-\int_0^t \mu_s dB_s - \frac{1}{2}\int_0^t \mu_s^2 ds\right)$$

Use Ito's lemma to compute dY and show that it is a (local) martingale. State the conditions that must be satisfied for Y_t to be well defined.

Note that we can write $M = e^Z$, where $dZ = -\frac{1}{2}\mu_t^2 dt - \mu_t dB_t$. Applying Ito's lemma

$$\frac{dM}{M} = -\mu_t dB$$

Now apply Ito's lemma to the Y process

$$dY = dM \cdot X + M \cdot dX + dM \cdot dX \quad \Rightarrow \quad \frac{dY}{M} = \frac{dM}{M} X + dX + \frac{dM}{M} dX$$

Substituting in for $\frac{dM}{M}$ and dX we get

$$dY = M(1 - \mu_t X) dB$$

This will be a local martingale if the integrand is in $\mathcal{L}^2$, i.e. $\int [M_s(1 - \mu_s X_s)]^2 ds < \infty$ w.p.1. It will be a global martingale if the integrand is in $\mathcal{H}^2$.

4. **Betting on a Brownian motion**. In class we discussed why ruling out arbitrage in continuous-time required restrictions on portfolio/betting strategies. This question considers a variant of the doubling strategy we studied. Let B_t be a Brownian motion, and define the stopping time $\tau = \inf\{t \geq 0 : B_t = 1\}$. We showed in class that $P[\tau < \infty] = 1$. Given this, consider the simple strategy of betting a *fixed* amount until $B_t = 1$. This also seems like a sure bet. Can you see any difference from the doubling strategy? Explain your reasoning both intuitively, and more formally using the distinction between martingales and local martingales. (Hint: Is it still the case here that $E_0[B_{\tau\wedge t}] = 1$?).

   With a fixed bet size $B_{\tau\wedge t}$ is not just a local martingale, but a global martingale as well. Hence, $E_0[B_{\tau\wedge t}] = 0$. Although the path hits 1 almost surely, its mean hitting time is infinite, so that small probabilities of very large losses keeps the mean 0. Increasing the size of the bet fights against the law of large numbers by compressing the effective time it takes to hit the target. Remember, changing the size of the bet is equivalent to increasing the variance of the process, which is equivalent to a time rescaling.

5. **The Feynman-Kac Formula**. There is a close relationship between 2nd-order partial differential equations and conditional expectations of diffusion processes. This relationship is revealed by the Feynman-Kac formula. The FK formula is widely used in financial economics.

   Consider the scalar diffusion process

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

   Suppose we want to compute the expected terminal payoff $G(X_T)$ at some future date T, given the current value of the process (e.g., we want to value an option)

$$g(x, t) = E[G(X_T)|X_t = x]$$

   Prove that g can be obtained by solving the following 2nd-order (linear) PDE

$$0 = g_t(x, t) + g_x(x, t)\mu(x, t) + \frac{1}{2}g_{xx}(x, t)\sigma^2(x, t)$$

   with boundary condition, $g(x, T) = G(x)$.

   (Hint: Use the law of iterated expectations to write

$$\begin{aligned} g(x, t) &= E[E[G(X_T)|X_{t+dt}]|X_t = x] \\ &= E[g(X_{t+dt}, t + dt)|X_t = x] \end{aligned}$$

   and then use Ito's lemma to write $g(X_{t+dt}, t + dt)$ in terms of $g(x, t)$ and its derivatives).

   Start with the approximation, $g(X_{t+dt}, t + dt) \approx g(x, t) + dg$. From Ito's lemma,

$$\begin{aligned} dg &= g_t dt + g_x dx + \frac{1}{2}\sigma^2 g_{xx} dt \\ &= g_t dt + g_x[\mu_t dt + \sigma_t dB_t] + \frac{1}{2}\sigma_t^2 g_{xx} dt \end{aligned}$$

   Therefore, $E[g(X_{t+dt}, t+dt)|X_t = x] = g(x, t)+g_t(x, t)+g_x(x, t)\mu(x, t)+\frac{1}{2}g_{xx}(x, t)\sigma^2(x, t)$. Cancelling the g term from both sides yields the result.

6. **Stochastic Growth with CARA utility**. In class we solved a stochastic growth model with CRRA utility. This problem asks you to solve a stochastic growth model with CARA preferences.

   Consider an agent who wants to solve the following problem

$$\max_c E \int_0^\infty e^{-\rho t} u(c) dt \qquad \text{where} \qquad u(c) = \frac{-1}{\gamma} e^{-\gamma c}$$

where the parameter γ is the coefficient of absolute risk aversion. The capital stock, k, evolves according to the following stochastic differential equation

$$dk = (\mu k - c) \cdot dt + \sigma dB$$

where dB is an increment to a Brownian motion process, and μ and σ are constant parameters.

(a) Write down the agent's (stationary) HJB equation.

The HJB equation is

$$\rho V = \max_c \left\{ \frac{-1}{\gamma} c^{-\gamma c} + (\mu k - c) \cdot V_k + \frac{1}{2}\sigma^2 V_{kk} \right\}$$

(b) Use a guess-and-verify strategy to solve the HJB equation. (Hint: Try the guess $V(k) = -\gamma^{-1} e^{Ak+B}$ where A and B are undetermined coefficients).

Using our guess, we can write the FOC for c as follows

$$e^{-\gamma c} = V_k = -\frac{A}{\gamma} e^{Ak+B}$$

Solving for c gives

$$c = -\frac{1}{\gamma} Ak - \frac{B}{\gamma} - \frac{1}{\gamma} \ln\left(\frac{-A}{\gamma}\right)$$

Substituting in for c in the HJB equation, and cancelling the common e^{Ak+B} *term gives*

$$-\frac{\rho}{\gamma} = \frac{A}{\gamma^2} + \left[\left(\mu + \frac{A}{\gamma}\right) k + \frac{B}{\gamma} + \frac{1}{\gamma} \ln\left(\frac{-A}{\gamma}\right)\right] \left(\frac{-A}{\gamma}\right) - \frac{1}{2}\sigma^2 \frac{A^2}{\gamma}$$

The term multiplying k must be zero, which implies $A = -\gamma\mu$. *Given this, we then get the following expression for B*

$$B = \frac{\mu - \rho}{\mu} + \frac{1}{2}\gamma\mu\sigma^2 - \ln(\mu)$$

(c) Given your answer to part (b), write down the agent's optimal consumption/savings policy. Interpret your answer in terms of intertemporal substitution and precautionary saving.

Given our solutions for A and B, we can then derive the following expression for the consumption policy function

$$c = \mu k + \frac{1}{\gamma\mu} \left[\rho - \mu - \frac{1}{2}\sigma^2\gamma^2\mu^2\right]$$

The $\rho - \mu$ *term reflects intertemporal substitution, while the last term reflects precautionary saving.*