NEW YORK UNIVERSITY
Department of Economics

Econ-GA 2021 | Prof. Kasa
Financial Economics I | Fall 2018

PROBLEM SET 3
(Due October 29)

Each of the following questions is worth 10 points.

1. **Stochastic Volatility**. In class we solved the Merton problem when the 'investment opportunity set' was constant (ie., μ and σ we constants). This question asks you to consider the case where volatility is stochastic. There is strong empirical evidence to support this. Hence, now suppose the risky asset price follows the process

$$\frac{dS}{S} = \mu dt + \sigma_t dB$$

where σ_t also follows a geometric Brownian motion process

$$d\sigma_t = \sigma_t dB^{\sigma}$$

For simplicity, suppose dB and dB^{σ} are uncorrelated. Finally, continue to assume the investor has time-additive CRRA preferences

$$V(W,\sigma) = \max_{c,\pi} E_0 \int_0^{\infty} \frac{C^{1-\gamma}}{1-\gamma} e^{-\delta t} dt$$

subject to $dW = [(r + \pi(\mu - r))W - C]dt + \pi\sigma_t W dB$.

(a) Write down the investor's stationary HJB equation.

(b) Verify that a solution is of the form $V(W,\sigma) = f(\sigma)W^{1-\gamma}$.

(c) Derive a 2nd-order ODE for $f(\sigma)$. Can you solve it? Under what parameter restrictions do you get an economically sensible result?

(d) Is the investor's optimal portfolio still time invariant? Why or why not?

2. **Time-Varying Expected Returns**. In class we solved the Merton problem when the 'investment opportunity set' was constant (ie., μ and σ we constants). This question asks you to consider the case where the mean return is stochastic. There is strong empirical evidence to support this. Hence, now suppose the risky asset price follows the process

$$\frac{dS}{S} = \mu_t dt + \sigma dB$$

where μ_t follows a mean-reverting Ornstein-Uhlenbeck process

$$d\mu_t = \alpha(\bar{\mu} - \mu_t)dt + \sigma dB^{\mu}$$

For simplicity, suppose dB and dB^{μ} are uncorrelated. Finally, continue to assume the investor has time-additive CRRA preferences

$$V(W,\mu) = \max_{c,\pi} E_0 \int_0^{\infty} \frac{C^{1-\gamma}}{1-\gamma} e^{-\delta t} dt$$

subject to $dW = [(r + \pi(\mu_t - r))W - C]dt + \pi\sigma W dB$.

(a) Write down the investor's stationary HJB equation.

(b) Verify that a solution is of the form $V(W, \mu) = g(\mu)W^{1-\gamma}$.

(c) Derive a 2nd-order ODE for $g(\mu)$. Can you solve it? Under what parameter restrictions do you get an economically sensible result?

(d) Is the investor's optimal portfolio still time invariant? Why or why not?

3. **Girsanov's Theorem and the Gordon Growth Model**. Consider an asset paying dividends D over an infinite horizon. Assume D follows a geometric Brownian motion,

$$\frac{dD}{D} = \mu dt + \sigma dB$$

where μ and σ are constant. Also assume the instantaneous riskless rate, r, is constant, and that there is an SDF process M such that

$$\left(\frac{dD}{D}\right)\left(\frac{dM}{M}\right) = -\sigma\lambda dt$$

where λ is constant, and $\mu - \sigma\lambda < r$.

(a) Using the present value relation, $P_t = E_t \int_t^\infty M_s D_s ds$, show that the asset price is

$$P_t = \frac{D_t}{r + \sigma\lambda - \mu}$$

(Note: we are assuming the absence of bubbles).

(b) Show that the asset's Sharpe ratio is λ.

(c) Assume that the SDF takes the form $M_t = e^{-\delta t}(C_t/C_0)^{-\gamma}$, where $C = D$. In this case, what is λ?

(d) Using Girsanov's Theorem, show that

$$\frac{dD}{D} = (\mu - \sigma\lambda)dt + \sigma d\tilde{B}$$

where $\tilde{B}$ is a Brownian motion under the risk-neutral Q-measure associated with M.

(e) Use the risk-neutral measure to show once again that $P_t = D_t/(r + \sigma\lambda - \mu)$.

4. **Two Trees**. Consider an economy with two independent Lucas trees,

$$\frac{dD_i}{D_i} = \mu_i dt + \sigma_i dB_i \qquad i = 1, 2$$

where B_1 and B_2 are uncorrelated, and μ_i and σ_i are constant. Aggregate consumption is given by $C_t = D_{1t} + D_{2t}$. Assume agents have log preferences, so the SDF process is given by

$$M_t = e^{-\delta t}\frac{C_0}{C_t}$$

(a) Define the state variable, $X_t = D_{1t}/C_t$. Show that the price of an equity claim to the dividends from Tree-1 is given by

$$P_{1t} = E_t \int_t^\infty \frac{M_s}{M_t} D_{1s} ds = f(X_t)C_t$$

for some function $f(X)$.

(b) Derive an ODE that characterizes $f(X)$.

(c) Explain intuitively why expected returns typically display 'momentum', and why an asset's price might change without any news about its dividends.

(For further details, see "Two Trees" by Cochrane, Longstaff, and Santa-Clara (*RFS*, 2008)).