NEW YORK UNIVERSITY
Department of Economics

Econon-GA 2021 | Prof. Kasa

Financial Economics I | Fall 2018

PROBLEM SET 3
(Solutions)

Each of the following questions is worth 10 points.

1. **Stochastic Volatility**. In class we solved the Merton problem when the 'investment opportunity set' was constant (ie., μ and σ we constants). This question asks you to consider the case where volatility is stochastic. There is strong empirical evidence to support this. Hence, now suppose the risky asset price follows the process

$$\frac{dS}{S} = \mu dt + \sigma_t dB$$

where σ_t also follows a geometric Brownian motion process

$$d\sigma_t = \sigma_t dB^\sigma$$

For simplicity, suppose dB and dB^σ are uncorrelated. Finally, continue to assume the investor has time-additive CRRA preferences

$$V(W,\sigma) = \max_{c,\pi} E_0 \int_0^\infty \frac{C^{1-\gamma}}{1-\gamma} e^{-\delta t} dt$$

subject to $dW = [(r + \pi(\mu - r))W - C]dt + \pi\sigma_t W dB$.

(a) Write down the investor's stationary HJB equation.

The HJB eq is

$$\delta V = \max_{c,\pi} \left\{ \frac{1}{1-\gamma} C^{1-\gamma} + [(r + \pi(\mu - r))W - C]V_w + \frac{1}{2}\pi^2\sigma_t^2 W^2 V_{ww} + \frac{1}{2}\sigma_t^2 V_{\sigma\sigma} \right\}$$

(b) Verify that a solution is of the form $V(W,\sigma) = f(\sigma)W^{1-\gamma}$.

Given the posited functional form for $V = f(\sigma)W^{1-\gamma}$ we find (note, we are implicitly assuming $\gamma \neq 1$)

$$C = [(1-\gamma)f]^{-1/\gamma} W \qquad \pi = \frac{\mu - r}{\gamma\sigma_t^2}$$

Subbing these back into the HJB eq we find that the terms involving C and W cancel out.

(c) Derive a 2nd-order ODE for $f(\sigma)$. Can you solve it? Under what parameter restrictions do you get an economically sensible result?

After cancelling out C and W we are left with the following ODE for $f(\sigma)$

$$(\delta - r)f(\sigma) = \frac{(\mu - r)^2(1+\gamma)}{2\gamma\sigma^2} f(\sigma) + \frac{1}{2}\sigma^2 f''(\sigma)$$

This has an 'analytic' solution in terms of the Bessel function, but it is not very illuminating. In practice, it would be more useful to solve it numerically, or to approximate the solution using a series expansion (note the equation is <u>linear</u>). From the solution for C, note that f must be of the form $(1-\gamma)^{-1}\tilde{f}(\sigma)$.

(d) Is the investor's optimal portfolio still time invariant? Why or why not?

The investor's portfolio, $\pi_t = \frac{\mu - r}{\gamma\sigma_t^2}$*, is no longer time invariant. It changes as* σ_t *changes. When volatility increases, the share invested in risky assets decreases (since volatility shifts are persistent).*

2. **Time-Varying Expected Returns**. In class we solved the Merton problem when the 'investment opportunity set' was constant (ie., μ and σ we constants). This question asks you to consider the case where the mean return is stochastic. There is strong empirical evidence to support this. Hence, now suppose the risky asset price follows the process

$$\frac{dS}{S} = \mu_t dt + \sigma dB$$

where μ_t follows a mean-reverting Ornstein-Uhlenbeck process

$$d\mu_t = \alpha(\bar{\mu} - \mu_t)dt + \sigma dB^{\mu}$$

For simplicity, suppose dB and dB^{μ} are uncorrelated. Finally, continue to assume the investor has time-additive CRRA preferences

$$V(W, \mu) = \max_{c,\pi} E_0 \int_0^{\infty} \frac{C^{1-\gamma}}{1-\gamma} e^{-\delta t} dt$$

subject to $dW = [(r + \pi(\mu_t - r))W - C]dt + \pi\sigma W dB$.

(a) Write down the investor's stationary HJB equation.

The HJB eq is

$$\delta V = \max_{c,\pi} \left\{ \frac{1}{1-\gamma} C^{1-\gamma} + [(r + \pi(\mu - r))W - C]V_w + \frac{1}{2}\pi^2\sigma^2 W^2 V_{ww} + [\alpha(\bar{\mu} - \mu_t)]V_{\mu} + \frac{1}{2}\sigma_{\mu}^2 V_{\mu\mu} \right\}$$

(b) Verify that a solution is of the form $V(W, \mu) = g(\mu)W^{1-\gamma}$.

Given the posited functional form for $V = g(\mu)W^{1-\gamma}$ *we find (note, we are implicitly assuming* $\gamma \neq 1$*)*

$$C = [(1-\gamma)g]^{-1/\gamma} W \qquad \pi = \frac{\mu_t - r}{\gamma\sigma^2}$$

Subbing these back into the HJB eq we find that the terms involving C *and* W *cancel out.*

(c) Derive a 2nd-order ODE for $g(\mu)$. Can you solve it? Under what parameter restrictions do you get an economically sensible result?

After cancelling out C *and* W *we are left with the following ODE for* $g(\mu)$

$$(\delta - r)g(\mu) = \frac{(\mu_t - r)^2(1+\gamma)}{2\gamma\sigma^2} g(\mu) + \alpha(\bar{\mu} - \mu)g'(\mu) + \frac{1}{2}\sigma_{\mu}^2 g''(\mu)$$

This has an 'analytic' solution in terms of the confluent hypergeometric function, but it is not very illuminating. In practice, it would be more useful to solve it numerically, or to approximate the solution using a series expansion (note the equation is linear). From the solution for C*, note that* g *must be of the form* $(1-\gamma)^{-1}\tilde{g}(\mu)$.

(d) Is the investor's optimal portfolio still time invariant? Why or why not?

The investor's portfolio, $\pi_t = \frac{\mu_t - r}{\gamma\sigma^2}$*, is no longer time invariant. It changes as* μ_t *changes. When expected returns increase, the share invested in risky assets increases (since mean return shifts are persistent)*

3. **Girsanov's Theorem and the Gordon Growth Model**. Consider an asset paying dividends D over an infinite horizon. Assume D follows a geometric Brownian motion,

$$\frac{dD}{D} = \mu dt + \sigma dB$$

where μ and σ are constant. Also assume the instantaneous riskless rate, r, is constant, and that there is an SDF process M such that

$$\left(\frac{dD}{D}\right)\left(\frac{dM}{M}\right) = -\sigma\lambda dt$$

where λ is constant, and $\mu - \sigma\lambda < r$.

(a) Using the present value relation, $P_t = E_t \int_t^\infty M_s D_s ds$, show that the asset price is

$$P_t = \frac{D_t}{r + \sigma\lambda - \mu}$$

(Note: we are assuming the absence of bubbles).

From the given information we know

$$\frac{dM}{M} = -rdt - \lambda dB$$

Solving, and normalzing $M_t = 1$, gives

$$M_s = e^{-(r+\frac{1}{2}\lambda^2)(s-t)-\lambda(B_s-B_t)}$$

From the given dividend process we know

$$D_s = D_t e^{(\mu-\frac{1}{2}\sigma^2)(s-t)+\sigma(B_s-B_t)}$$

Therefore

$$\begin{aligned} P_t &= D_t E_t \int_t^\infty e^{-(r+\frac{1}{2}\lambda^2)(s-t)-\lambda(B_s-B_t)} e^{(\mu-\frac{1}{2}\sigma^2)(s-t)+\sigma(B_s-B_t)} ds \\ &= D_t \int_t^\infty e^{-(r+\frac{1}{2}\lambda^2+\frac{1}{2}\sigma^2-\mu)(s-t)+\frac{1}{2}(\sigma-\lambda)^2(s-t)} ds \\ &= \frac{D_t}{r+\sigma\lambda-\mu} \end{aligned}$$

(b) Show that the asset's Sharpe ratio is λ.

From the result in (a) we know

$$\frac{dP}{P} = \frac{dD}{D} = \mu dt + \sigma dB$$

Therefore

$$\mu_S = \frac{D + dP}{P} = r + \sigma\lambda - \mu + \mu = r + \sigma\lambda$$

We also know

$$\sigma_S = \sigma$$

Therefore the Sharpe ratio is

$$\text{Sharpe} = \frac{\mu_S - r}{\sigma_S} = \frac{\sigma\lambda}{\sigma} = \lambda$$

(c) Assume that the SDF takes the form $M_t = e^{-\delta t}(C_t/C_0)^{-\gamma}$, where $C = D$. In this case, what is λ?

We know that λ *is (minus) the diffusion coefficient on* $\frac{dM}{M}$, *which is (minus) the diffusion coefficient on* $d\log(M)$, *which is (minus) the diffusion coefficient on* $-\gamma d\log(D_t)$, *which is just* $\gamma\sigma$.

(d) Using Girsanov's Theorem, show that

$$\frac{dD}{D} = (\mu - \sigma\lambda)dt + \sigma d\tilde{B}$$

where $\tilde{B}$ is a Brownian motion under the risk-neutral Q-measure associated with M.

With respect to the risk-neutral Q*-measure, we know that*

$$d\hat{B} = dB + \lambda dt$$

is a Brownian motion, where $\lambda =$ *the price of risk. Hence, changing measures from* P *to* Q *just involves replace* dB *with* $d\hat{B} - \lambda dt$. *Making the substitution gives*

$$\frac{dD}{D} = (\mu - \sigma\lambda)dt + \sigma d\tilde{B}$$

(e) Use the risk-neutral measure to show once again that $P_t = D_t/(r + \sigma\lambda - \mu)$.

Using the risk-neutral Q*-measure, the asset price is just the expected discounted value of dividends, using the risk-free rate to discount*

$$\begin{aligned} P_t &= E_t^Q \int_t^\infty e^{-r(s-t)} D_s ds \\ &= D_t \int_t^\infty e^{-r(s-t)} e^{(\mu-\sigma\lambda-\frac{1}{2}\sigma^2)(s-t)+\frac{1}{2}\sigma^2(s-t)} ds \\ &= D_t \int_t^\infty e^{-(r+\sigma\lambda-\mu)(s-t)} ds = \frac{D_t}{r+\sigma\lambda-\mu} \end{aligned}$$

4. **Two Trees**. Consider an economy with two independent Lucas trees,

$$\frac{dD_i}{D_i} = \mu_i dt + \sigma_i dB_i \qquad i = 1, 2$$

where B_1 and B_2 are uncorrelated, and μ_i and σ_i are constant. Aggregate consumption is given by $C_t = D_{1t} + D_{2t}$. Assume agents have log preferences, so the SDF process is given by

$$M_t = e^{-\delta t}\frac{C_0}{C_t}$$

(a) Define the state variable, $X_t = D_{1t}/C_t$. Show that the price of an equity claim to the dividends from Tree-1 is given by

$$P_{1t} = E_t \int_t^\infty \frac{M_s}{M_t} D_{1s} ds = f(X_t)C_t$$

for some function $f(X)$.

Note that we have

$$\frac{M_s}{M_t} = e^{-\delta(s-t)}\frac{C_t}{C_s}$$

Imposing the market-clearing conditions $C_i = D_i$ *in the present value formula then gives*

$$\frac{P_{1t}}{D_t} = E_t \int_t^\infty e^{-\delta(s-t)} \frac{D_{1s}}{D_{1s} + D_{2s}} ds = E_t \int_t^\infty e^{-\delta(s-t)} X_s ds = f(X_t)$$

where the last equality follows since X *is Markov.*

(b) Derive an ODE that characterizes $f(X)$.

Applying Leibniz' Rule, note that f satisfies the differential equation

$$E[df] = (\delta f - X)dt$$

From Ito's lemma we know

$$E[df] = f'(X)E(dX) + \frac{1}{2}f''(X)E(dX)^2$$

All that remains is to calculate dX. This again follows from Ito's lemma. Define $X = g(D_1, D_2) = D_1/(D_1 + D_2)$

$$\begin{aligned} dX &= g_{D_1}dD_1 + g_{D_2}dD_2 + \frac{1}{2}g_{D_1D_1}(dD_1)^2 + \frac{1}{2}g_{D_2D_2}(dD_2)^2 \\ &= X(1-X)[(\mu_1 - \mu_2) + (1-X)\sigma_2^2 - X\sigma_1^2]dt + X(1-X)[\sigma_1 dB_1 - \sigma_2 dB_2] \end{aligned}$$

From this we can easily calculate $E(dX)$ (it's just the drift of the above expression) and $E(dX)^2$ (it's just the sum of the squared diffusion coefficients). Subbing these in gives us an ODE for $f(X)$.

(c) Explain intuitively why expected returns typically display 'momentum', and why an asset's price might change without any news about its dividends.

With more than one Lucas tree, portfolio rebalancing becomes an issue. When tree 1 experiences a positive dividend shock, it's price rises. As a result it becomes a larger part of the market portfolio and its (absolute) covariance with the SDF process increases. This raises the risk premium on tree 1, which leads to higher expected returns (momentum). Note that the change in the SDF process will also trigger changes in the price of tree 2, even though there is no news about its dividends.

(For further details, see "Two Trees" by Cochrane, Longstaff, and Santa-Clara (*RFS*, 2008)).