Topics for Today

1.) The Arrow-Debreu General Equilibrium Model
   - The meaning of "complete markets"

2.) The "Anything Goes" Theorem
   - Sonnenschein (JET, 1973), Mantel (JET, 1974), Debreu (1974)

3.) The 2 Fundamental Theorems of Welfare Economics

4.) Time-0 Implementation of A-D Competitive Equil.

5.) Pareto Optimality
   - The correspondence between PO allocations and competitive equilibria

6.) Risk Sharing with Complete Markets

Arrow - Debreu

- The AD model is an important theoretical benchmark. Most recent work in macro attempts to remedy its limitations.

- There are 2 key ideas in AD:

1. Convert dynamic, stochastic settings into standard (static) general equilibrium theory by indexing goods by date & state (i.e., "state- and date- contingent claims").

Limitations

a.) Must have complete markets. If there are T periods, S states in each period, and N goods, then there must be $S \times N \times T$ markets!

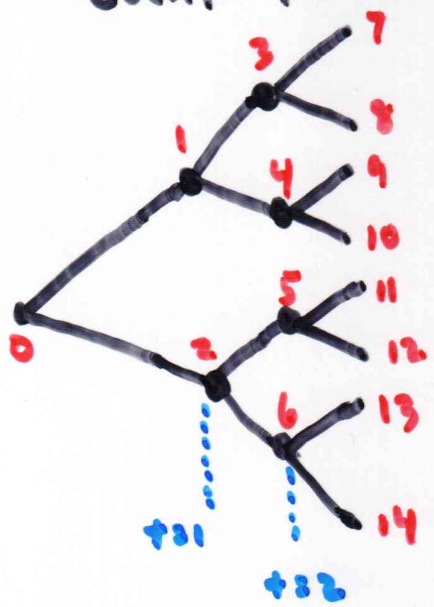
b.) All trades take place at time-0, before the resolution of uncertainty. (Deliveries vs. Trades). This really pushes the Walrasian auctioneer idea to the extreme!

- To a certain extent, both these limitations are addressed by the 2nd key idea of AD:

(2.) Can relax the time-0 trading assumption, and greatly reduce the number of required markets by allowing dynamic trading of "Arrow securities".

Arrow Security: A security that pays off one unit of the numeraire if (and only if) a given state is realized next period.

- To see this, suppose 1-period ahead uncertainty is binary (up or down). We have the following "event tree"



Standard AD ⟹ Need 15 (time-0) markets

with Arrow securities need only 2!

S_u : pays 1 iff next period's state is up

S_d : pays 1 iff next period's state is down

Fact 1: Arrow securities can be interpreted as an orthogonal basis for the 1-step ahead state space. (Any other linearly independent set of securities will work).

Fact 2: The Wiener Process (Brownian Motion) is the continuous-time limit of a binomial tree.

Implication: If the underlying uncertainty evolves as a Brownian motion, continuous trading of just 2 (independent) securities (e.g., stock + bond) span the entire state space! All other securities are "redundant", and can be priced by no arbitrage.

This is a key idea in finance!

Black + Scholes (JPE, 1973)

Harrison + Kreps (JET, 1979)

Duffie + Huang (Econometrica, 1986).

Comments on AD

1.) Knowledge. The relevant state space is known, and realizations of the state are verifiable. AD does not easily handle asymmetric info. or "Rumsfeld uncertainty" (unknown unknowns).

2.) Commitment. No reneging/No default. Ex post, after uncertainty is realized, an agent may not want to honor his AD contracts. It is assumed that AD contracts can be enforced. (Note: Since all trades occur at time-0, with known prices, unintentional default is not an issue).

3.) Dynamic Consistency. As we shall see, time-0 trading in a full menu of AD state-contingent claims produces a Pareto optimal allocation. Hence, if markets were to re-open in the future, no trades would occur. (Some person might want to trade, but if so, there would be no one else on the 'other side' of the market.

4.) Market Power: All agents must be competitive (i.e., price-takers).

5.) No Money. There is no role for money in AD general equilibrium (other than as a unit of account).

The Sonnenschein-Mantel-Debreu Theorem (Anything Goes)

- At this point, at least in terms of descriptive realism, it would appear that AD general equil. theory is absurd. Somewhat surprisingly, it places very few restrictions on aggregate data. That is, based on aggregate data alone, it is very hard to reject complete markets!

- S-M-D addressed the following question:

  What restrictions are imposed on aggregate excess demand functions, $Z(p)$, by the General Equil. model?

ANSWER: 1.) Continuity

2.) Homogeneity of degree zero $[Z(\lambda p) = Z(p)]$

3.) Walras' Law $[p \cdot Z(p) = 0]$.

- In other words, give me a function that satifies these 3 restrictions, and I can construct an economy that has this function as its equilibrium.

- Hence, to reject AD we need to look at micro/cross-sectional data.

The 2 Fundamental Theorems of Welfare Economics

- Now let's turn to the normative/welfare implications of the AD model:

First Welfare Theorem : With complete markets, competitive equilibria are Pareto Optimal.

Comments

1.) Externalities are a form of incomplete markets. Hence, the theorem rules out (non-pecuniary) externalities.

2.) By definition, competitive equilibrium means all agents act as price-takers. Hence, the theorem rules out monopoly/monopsony power.

3.) It is important to remember that Pareto Optimality is a weak efficiency criterion. It ignores distributional issues.

4.) This theorem is relatively easy to prove (e.g., far easier than the proof of the existence of a competitive equilibrium).

Second Welfare Theorem: If preferences & technology are convex, any PO allocation can be supported (decentralized) as a Comp. Equil. with appropriate initial redistributions of resources.

Comments

1.) This theorem can be used as a computational short-cut. Rather than solve a complex Fixed Point problem (i.e., find prices that simultaneously clear all markets), we can compute an equil. in two steps: (1) Solve an optimization problem to find equil. quantities, and (2) Plug these quantities into the appropriate marginal conditions to find equil. prices. It's not much of an exaggeration to say that this short-cut is the reason why so many macroeconomists are reluctant to abandon complete markets!

2.) The 2nd welfare is harder to prove. Convexity is required because the proof appeals to the Separating Hyperplane Theorem.

Competitive Equilibrium

- Assume the state evolves according to a Markov process: $\text{Prob}[s_{t+1} = s' \mid s_t = s] = \pi(s' \mid s)$.

- Let $s^t = [s_0, s_1, \cdots s_{t-1}, s_t]$ "History up to t"

$$\pi(s^t) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \cdots \pi(s_1 \mid s_0)\pi(s_0)$$

- Assume $\exists$ I agents, $i = 1, 2, \cdots I$. Agent i owns a stochastic endowment process,

$$y_t^i = y^i(s_t)$$

Preferences

$$U(c^i) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)]\pi(s^t \mid s_0)$$

Key Variable

$q_t^0(s^t)$ = Time-0 price of a claim to time-t consumption, contingent on history s^t.

Budget Constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y^i(s_t)$$

Note: There is just one budget constraint.

Definitions

1.) Price system : A sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$

2.) Allocation : A vector of sequences of functions, $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ $\forall i$.

3.) Feasible Allocation : $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$ $\forall t$

4.) Competitive Equilibrium: A feasible allocation and a price system such that the allocation solves each agent's optimization problem.

FOCs

$$\beta^t u'[c_t^i(s^t)]\pi(s^t|s_0) = \mu^i q_t^0(s^t) \quad \forall i, \forall t$$

Lag. Multiplier
on budget constraint

Take ratio across agents,

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^j(s^t)]} = \frac{\mu^i}{\mu^j}$$

$$\Rightarrow c_t^i(s^t) = u'^{-1}\left\{u'[c_t^1(s^t)]\frac{\mu^i}{\mu^1}\right\} \quad \forall i$$

Sub into aggregate resource constraint,

$$\sum_i u'^{-1}\left\{u'[c_t^1(s^t)]\frac{\mu^i}{\mu^1}\right\} = \sum_i y^i(s_t)$$

history independent

Proposition: Competitive equilibrium allocations are not history dependent. ($c_t^i(s^t) = c_\tau^i(s^\tau)$ for s^t and s^τ s.t. $\sum_i y^i(s^t) = \sum_i y^i(s^\tau)$).

This depends on: (1) Complete Mkts., (2) Time separable preferences.

Example

Suppose $u^i(c) = \frac{1}{1-\gamma}(c^i)^{1-\gamma} \quad \forall i$

Then $(c_t^i)^{-\gamma} = (c_t^j)^{-\gamma}\left(\frac{\mu_i}{\mu_j}\right)$

$$\Longrightarrow \quad c_t^i = c_t^j \underbrace{\left(\frac{\mu_i}{\mu_j}\right)^{-1/\gamma}}_{\text{constant}}$$

$\Longrightarrow$ Consumption is perfectly correlated

1.) across agents.

2.) Individual consumption only depends on aggregate endowment.

$\Longrightarrow$ Imperfect cross-sectional correlation, or idiosyncratic variation over time is indicative of incomplete mkts. [Or is it? What happened to Sonnenschein-Mantel-Debreu?]

Pareto Problem

- Now seek an efficient outcome.

$$W = \sum_{i=1}^{I} \lambda_i U(c^i) \qquad \lambda_i = \text{time-invariant Pareto weight}$$

$$\text{s.t.} \quad \sum_i c_t^i(s^t) \le \sum_i y^i(s^t) \quad \forall t \text{ and } s^t$$

$$\boxed{\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u[c_t^i(s^t)] \pi(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y^i(s^t) - c_t^i(s^t)] \right\}}$$

FOCs

$$\lambda_i \beta^t u'[c_t^i(s^t)] \pi(s^t) = \theta_t(s^t) \quad \forall i, t, s^t$$

Take Ratio,

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^1(s^t)]} = \frac{\lambda_1}{\lambda_i}$$

$$\Rightarrow c_t^i(s^t) = u'^{-1}\left\{ u'[c_t^1(s^t)] \frac{\lambda_1}{\lambda_i} \right\}$$

$$\Rightarrow \boxed{\text{Same as before with } \lambda_i = \mu_i^{-1} \; !}$$

Note : Constant Pareto Weights $\Longleftrightarrow$ Constant Shares
With frictions λ_i are time-varying

Risk-Sharing

With complete markets, the 2nd Welfare Theorem applies, and we can interpret a competitive equilibrium as the outcome of Pareto (or social planning) problem. This is a useful perspective when studying risk sharing.

Risk-sharing is usually analyzed with HARA utility functions, since they deliver linear sharing rules. [Wilson (1968) - "The Theory of Syndicates"]

Hyperbolic Absolute Risk Aversion

$$\boxed{\underline{\text{HARA}}: \; u(c) = \alpha \frac{\gamma}{1-\gamma}\left(\beta + \frac{c}{\gamma}\right)^{1-\gamma}}$$

1.) $\lim_{\gamma \to \infty} \Longrightarrow u(c) = -\beta e^{-c/\beta}$ } constant absolute risk aversion

2.) $\beta = 0 \Longrightarrow u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ } constant relative risk aversion

3.) $\gamma = -1 \Longrightarrow u(c) = -\frac{1}{2}(\beta - c)^2 \quad \beta > c$

The HARA class is useful since it features linear risk tolerance [Risk Tolerance $= -\frac{u'}{u''}$]

Examples of Sharing Rules

Define $Y^w(s) = \sum_{i=1}^{N} y_i(s)$ $\Big\}$ Total world endowment in State s (N countries)

Pareto weights

~~1.) 2 countries~~ Pareto Problem: $\max_{c_i(s)} \sum_{i=1}^{N} \lambda_i \left\{ \sum_{s=1}^{S} \pi(s)\, u(c_i(s)) \right\}$

subject to $\sum_{i=1}^{N} c_i(s) = Y^w(s) \quad \forall s$

1.) 2 countries, Identical exponential, $u = -\frac{1}{\sigma} e^{-\sigma c}$

$$c_1(s) = \frac{1}{2} Y^w(s) + \frac{1}{\sigma}\frac{1}{2}\log\left(\frac{\lambda_1}{\lambda_2}\right)$$

$$c_2(s) = \frac{1}{2} Y^w(s) - \frac{1}{\sigma}\frac{1}{2}\log\left(\frac{\lambda_1}{\lambda_2}\right)$$

2.) Identical CRRA, $u = \frac{1}{1-\rho} c^{1-\rho}$

$$c_1(s) = \phi\, Y^w(s)$$

$$c_2(s) = (1-\phi)\, Y^w(s)$$

where $\phi = \dfrac{\lambda_1^{1/\rho}}{\lambda_1^{1/\rho} + \lambda_2^{1/\rho}}$