

## Topics for Today

- 1.) Risk Sharing (from last time)
- 2.) A Couple of Examples
- 3.) Pricing Derivative Securities
- 4.) Sequential Implementation of AD
- 5.) Recursive Implementation of AD
- 6.) A recursive Pareto Problem
  - "Dynamic Programming Squared"

## Example 1

Consider the following 2-agent/1-good economy:

### Preferences

Identical preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

### Endowments

Endowments are deterministic

$$y_t^1 = 1, 0, 0, 1, 0, 0, 1, \dots$$

$$y_t^2 = 0, 1, 1, 0, 1, 1, 0, \dots$$

Compute AD prices + allocations

### FOCs

$$q_0^0 = \frac{\beta^t u'(c_t^i)}{u'(c_0^i)}$$

Note:  $q_0^0 = 1$   
(prices in units of  
time-0 goods)

## Allocations

Note: No aggregate risk or variability.  $Y_t(s^*) = 1 \forall t$

$\Rightarrow$  with complete mkt., guess  $c_t^1 = \phi$   $c_t^2 = 1 - \phi$

$\Rightarrow$  From FOEs,  $q_t^0 = \beta^t$

To compute  $\phi$ , use budget constraints

$$\frac{\phi}{1-\beta} = \sum_{t=0}^{\infty} \beta^t y_t^1 = 1 + \beta^3 + \beta^6 + \dots$$
$$= \frac{1}{1-\beta^3}$$

$$\Rightarrow \phi = \frac{1-\beta}{1-\beta^3}$$

## Example 2

- Let's now consider an economy with aggregate risk
- Again assume 2-agents/1-good, with identical preferences:

### Preferences

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \ln[c^i_t(s^t)]$$

### Endowments

$$y^1_+ = s_+$$

$$y^2_+ = 1$$

where  $s_+$  follows a 2-state Markov chain with  $s_1 = 0$  and  $s_2 = 1$ , and with time-invariant transition probabilities,  $\pi(s_{t+1} = s' | s_t = s) = \pi(s' | s)$ .

- Let  $Y(s_+) = y^1_+ + y^2_+$  be the aggregate endowment

Question: What are the AD prices + allocations ?

- Since 3 complete mkt., let's compute allocations using the 2<sup>nd</sup> Welfare Theorem (by solving a Pareto Problem). Since preferences are time-separable & the good is non-storable, we can just consider 1 period

$$\max_{c^1, c^2} \{ \theta \ln(c_+^1) + (1-\theta) \ln(c_+^2) \}$$

s.t.  $c_+^1 + c_+^2 = Y_+$

FOC (with resource constraint subbed in)

$$\frac{\theta}{c_+^1} = \frac{1-\theta}{Y_+ - c_+^1}$$

$$\Rightarrow c_+^1 = \theta Y_+ \quad c_+^2 = (1-\theta) Y_+$$

$$q_+^0 = \frac{\beta^+ \pi(s^+ | s_0) u'(c_+^1)}{u'(c_+^1)} = \frac{\beta^+ \pi(s^+ | s_0) c_+^1(s_0)}{c_+^1(s^+ | s_0)}$$

$$\Rightarrow q_+^0 = \frac{\beta^+ \pi(s^+ | s_0) Y(s_0)}{Y_+(s^+)}$$

Note: Date/state contingent claim price higher when:

- 1.) Short maturity
  - 2.) State is likely to occur
  - 3.) Aggregate endowment is (relatively) low in that state
- Finally, to find equilibrium value of  $\theta$ , use  $q_t^0$  and  $\Pi(s'|s)$  to evaluate PDV of consumption and endowments, and impose budget balance.

### Questions

- 1.) Conditional on being in state 1 at time-0 ( $s_0 = 0$ ), what is the price of a claim to state-2/date-2 output?

$$q_2^0(s_2=2|s_0=1) = \beta^2 [\pi_{11} \pi_{12} + \pi_{12} \pi_{22}] \frac{Y_0}{Y_2}$$
$$= \beta^2 [\pi_{11} \pi_{12} + \pi_{12} \pi_{22}]^{1/2}$$

- 2.) Again conditional on  $s_0 = 0$ , what is the price of a claim to state-1/date-2 output?

$$q_2^0(s_2=1|s_0=1) = \beta^2 [\pi_{11} \pi_{11} + \pi_{12} \pi_{21}] \frac{Y_0}{Y_2}$$
$$= \beta^2 [\pi_{11}^2 + \pi_{12} \pi_{21}] \cdot 1$$

3.) What is the price of a sure claim to date-2 consumption (conditional on  $s_0 = 0$ )?

4.) What if ~~one~~ Agent 1 had linear preferences?

$$c_t^1 = Y_t - \phi$$

$$c_t^2 = \phi$$

5.) What if  $u^1 = -\frac{1}{\sigma_1} e^{-\sigma_1 c^1}$ ,  $u^2 = -\frac{1}{\sigma_2} e^{-\sigma_2 c^2}$

$$c_t^1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} Y_t + \text{constant}$$

$$c_t^2 = \frac{\sigma_1}{\sigma_1 + \sigma_2} Y_t - \text{constant}$$

## Pricing Derivative Securities

- As long as mkt. are complete, the above examples are quite general.  $q_f^o$  can be used to price any derivative security.

Example 1: Riskless (real) consol. (Claim to 1 unit of consumption forever).

$$\sum_{t=0}^{\infty} \sum_{S^t} q_f^o(S^t)$$

Example 2: An arbitrary stream of history-contingent payoffs,  $d_t(S^t)$ , where  $d_t(S^t)$  is any (measurable) function of  $S^t$ .

$$\sum_{t=0}^{\infty} \sum_{S^t} q_f^o(S^t) d_t(S^t)$$

- If these pricing relations do not hold, then there are arbitrage opportunities.

Question: What if mkt. are incomplete?

$q_f^o$  will in general be non-unique

## Sequential Implementation of AD

- Time-0 AD requires a lot of markets. Descriptively, it is absurd. However, we'll now see that the same allocations can be supported with far fewer markets, as long as agents can dynamically trade a complete set of "Arrow securities".
- We'll consider 2 cases : (1) Sequential, and (2) Recursive. The sequential case is less restrictive. It's useful because it shows that the dynamic spanning idea applies even when there is complex history-dependence in the economy (preferences, endowments, state probabilities). However, we'll then see that if the economy itself has a recursive (stationary/Markov) structure, then so does the sequential AD equilibrium.
- The key idea is to allow trading in a set of one-period ahead state-contingent claims:

$Q_t(s_{t+1} | s^t)$  = price of one unit of time- $(t+1)$  consumption, contingent on the state at  $t+1$  being  $s_{t+1}$ , given the history  $s^t$

- Next, we need to define a state variable that captures, for each date + history, the current (net) claims to consumption of each agent:

$a_t^i(s^+)$  = claims to time- $t$ /history- $s^+$  consumption held by agent- $i$  at time- $t$  (net of his own endowment).

(Note: In equil.,  $\sum_i a_t^i(s^+) = 0 \quad \forall t$ ).

- Using this we can decompose the time-0 AD budget constraint into a sequence of budget constraints:

$$c_t^i(s^+) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^+) Q(s_{t+1} | s^+) \leq y_t^i(s^+) + a_t^i(s^+)$$

- Finally, when allowing agents to trade over time, we need to rule out Ponzi schemes. Let,

$A_t^i(s^+)$  = value of agent  $i$ 's future endowment, given history  $s^+$

$$= \sum_{\tau=t}^{\infty} \sum_{s^{\tau} | s^+} q_{f^{\tau}}^+(s^{\tau}) y_{\tau}^i(s^{\tau})$$

- Then we impose

$$-a_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}) \quad \forall t$$

If this constraint ever binds, then the only way the agent can honor his current obligations is by setting consumption to zero, always & forevermore

- Let  $\eta_+^i(s^+) = \text{Lagrange Mult. on budget constraint}$   
 $\eta_+^i(s^+; s_{++1}) = \text{Lagrange Mult. on "debt constraint".}$

### FOCs

$$c_+^i(s^+): \quad \beta^+ \pi_+(s^+) u'(c_+^i(s^+)) - \eta_+^i(s^+) = 0$$

$$a_{t+1}^i(s_{++1}, s^+): \quad -\eta_+^i(s^+) Q_+(s_{++1}|s^+) + v_+^i(s^+; s_{++1}) + \eta_{t+1}^i(s_{++1}, s^+) = 0$$

- Note, if preferences satisfy an "Inada Condition" (i.e.,  $u'(0) = \infty$ ), then  $v_+^i = 0$ . (Why?). Imposing this, and combining the FOCs:

$$Q_+(s_{++1}|s^+) = \beta \pi_+(s^{++1}|s^+) \frac{u'(c_{++1}(s^{++1}))}{u'(c_+(s^+))}$$

- Now go back to the Time-0 AD FOCs. Take ratio of dates  $t+1$  and  $t$ :

$$\frac{\beta \pi(s^{t+1}|s^t) u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))} = \frac{q_{t+1}^o(s^{t+1})}{q_t^o(s^t)}$$

Hence, if  $Q(s_{t+1}|s^t) = \frac{q_{t+1}^o(s^{t+1})}{q_t^o(s^t)}$ , the two market structures deliver the same allocations!

### Loose End

The sequential equilibrium must start with an initial condition, i.e., a given value of  $a_0^i(s_0)$ . These are "free parameters". What should they be if we are to get identical allocations? Intuitively, it should be the case that

$$a_0^i(s_0) = 0 \quad \forall i.$$

Sketch of Proof: Let  $\tilde{c}_t^i(s^t)$  = Time-0 AD allocation  
 $w_t^i(s^t) = \sum_{\tau=1}^{\infty} \sum_{s^{\tau} \mid s^t} q_{\tau}^t(s^{\tau}) [\tilde{c}_{\tau}^i(s^{\tau}) - y_{\tau}^i(s^{\tau})]$   
= current + future net claims in Time-0 allocation

Guess & Verify that  $a_{t+1}^i(s_{t+1}, s^t) = w_{t+1}^i(s^{t+1})$  when  $a_0^i(s_0) = 0$

## Recursive Implementation of AD

- Although sequential implementation with Arrow securities achieves a dramatic reduction in the number of required markets, in general it could still be quite complicated.
- For example,  $Q_+(S_{t+1} | S^*)$  could exhibit complicated variation over time as a function of the history  $S^*$ . Same thing with allocations.
- However, if we impose the following (and continue to assume time-separable preferences):

1.) The exogenous state is stationary/Markov

$$\pi_+(s^*) = \pi(s_+ | s_{+-}) \cdot \pi(s_{+-} | s_{-+}) \cdot \dots \cdot \pi(s_0)$$

2.) Endowments are time invariant functions of the current state

$$y_i(s^*) = y_i(s_+)$$

Then the agent's problem is recursive, and we can characterize the AD equil. using Bellmen Eqs.

## Bellman Equation

$$V^i(a, s) = \max_{c^i, \hat{a}^i(s')} \left\{ u(c^i) + \beta \sum_{s'} V^i(\hat{a}^i(s'), s') \pi(s'|s) \right\}$$

s.t. 1.)  $c^i + \sum_{s'} \hat{a}^i(s') Q(s'|s) \leq y^i(s) + \alpha^i$

2.)  $\hat{a}^i(s') \leq A^i(s') \quad \forall s'$

## Policy Functions

$$c^i = h^i(a^i, s)$$

$$\hat{a}^i(s') = g^i(a^i, s, s')$$

## Equil. Arrow Security Prices

$$Q(s_{t+1}|s_t) = \frac{\beta \pi(s_{t+1}|s_t) u'(c_{t+1}^i)}{u'(c_t^i)}$$

## Recursive Pareto

2 agents / No Aggregate Uncertainty

$$y_t^1 = s_t \quad y_t^2 = 1 - s_t$$

$$s_t \sim \text{i.i.d.} \Rightarrow \Pi_t(s^t) = \Pi(s_t) \Pi(s_{t-1}) \dots \Pi(s_0)$$

$s_t$  is discrete,  $s_t \in [\bar{s}_1, \bar{s}_2, \dots, \bar{s}_S]$ ,  $\bar{s}_{i+1} > \bar{s}_i$

$$\Pi_i = \text{Prob}[s_t = \bar{s}_i].$$

Planner does 2 things at each date:

- 1.) Delivers a current allocation
- 2.) Promises future discounted utility streams

Let,

$v$ : Expected PDV of utility for agent 1

$P(v)$ : Maximum expected PDV of agent 2's utility,  
given that agent 1 is promised  $v$ .

- $V$  and  $P(V)$  can be written recursively,

$$V = \sum_{i=1}^s \pi_i [u(c_i) + \beta w_i]$$

$$P(V) = \sum_{i=1}^s \pi_i [u(1-c_i) + \beta P(w_i)]$$

Feasible Set:  $V = [\frac{u(\varepsilon)}{1-\beta}, \frac{u(1)}{1-\beta}]$

- We can now write the following recursive Pareto problem

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$$P(V) = \max_{c_i, w_i} \sum_{i=1}^s \pi_i [u(1-c_i) + \beta P(w_i)]$$

s.t.  $\sum_{i=1}^s \pi_i [u(c_i) + \beta w_i] \geq V$

$$w_i \in V$$

$$c_i \in [0, 1]$$

*Promise keeping constraint*

*"Dynamic Programming Squared"*

## FOCs

Lagrange Mult. on Promise  
Keeping constraint

$$c_i: -u'(1-c_i) + \theta u'(c_i) = 0$$

$$w_i: p'(w_i) + \theta = 0$$

$$\text{Envelope} \Rightarrow p'(v) = -\theta$$

$$\Rightarrow p'(w_i) = p'(v)$$

$$\Rightarrow w_i = v$$

$\Rightarrow$  continuation utility  
is state independent

From the first FOC

$$\frac{u'(1-c_i)}{u'(c_i)} = \theta = -p'(v) \quad \left. \begin{array}{l} \text{Relative Pareto} \\ \text{weight for} \\ \text{Agent } i \end{array} \right\}$$

$\Rightarrow$  consumption allocations are constant  
across time and states

$\Rightarrow$  complete risk sharing