Topics for Today

1.) Basic Growth Facts (from last time)

2.) Alternative Growth Models

3.) Problems with Solow

4.) The Cass-Koopmans Model
- The Planner's Problem
- Phase Diagram + Saddlepath
- Linearization/Local Convergence Rates

Leading Growth Models

1.) Solow Model
- everything exogenous (saving, labor, technology)

2.) Cass-Koopmans Model
- saving endogenous; labor + technology exogenous

3.) Diamond Model
- same as Cass-Koopmans except finite-lived OLG (incomplete mkts).

exogenous growth

4.) Romer (JPE, 1986)
- externalities

5.) Lucas (JME, 1988)
- human capital accumulation

6.) Romer (JPE, 1990)
- R+D.

endogenous growth

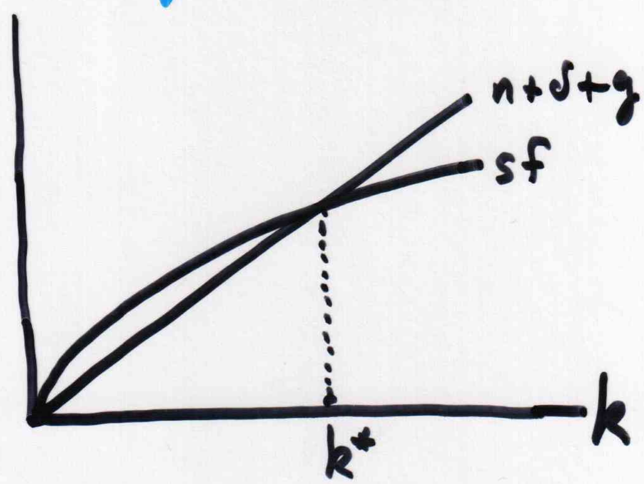
Solow

$$Y = K^{\alpha}(AL)^{1-\alpha} \qquad \left.\frac{\dot{A}}{A} = g \right\} \text{Labor-augmenting tech. progress}$$

$$k = \frac{K}{LA} \qquad \frac{\dot{L}}{L} = n$$

$$\boxed{\dot{k} = s f(k) - (n+\delta+g)k}$$

saving rate



Problems with Solow

1.) Cannot explain long-run per capita growth.
- Tech. assumed exogenous!
- Changes in saving + investment affect steady state level of output, but not its growth rate

2.) Differences in savings rates cannot explain differences in international income levels.

Example

Suppose rich & poor have same technology.

$$y_r = Ak_r^{\alpha} \quad y_p = Ak_p^{\alpha} \implies \frac{y_r}{y_p} = \left(\frac{k_r}{k_p}\right)^{1/3}$$

$$\alpha \approx 1/3$$

$$\implies \frac{k_r}{k_p} = \left(\frac{y_r}{y_p}\right)^3$$

$$\frac{y_r}{y_p} = 10 \implies \frac{k_r}{k_p} = 1000 \, !$$

Or, since $\frac{k_r}{k_p} = \left(\frac{s_r}{s_p}\right)^{\frac{1}{1-\alpha}} \implies \frac{y_r}{y_p} = \left(\frac{s_r}{s_p}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{s_r}{s_p}\right)^{1/2}$

Therefore, $\frac{s_r}{s_p} = \left(\frac{y_r}{y_p}\right)^2 = 100 \, !$

Or, since $r = \alpha k^{\alpha-1}$, $\frac{r_p}{r_r} = \left(\frac{k_p}{k_r}\right)^{\alpha-1} = \left(\frac{y_r}{y_p}\right)^{\frac{1-\alpha}{\alpha}}$

$$\implies \frac{r_p}{r_r} = \left(\frac{y_r}{y_p}\right)^2 = 100 \, !$$

3.) Predicted Convergence Rates Too High

Example

Assume $g = 0$

Linearize the Solow diff. eq. (around steady state)

$$\dot{k} = s\alpha (k^*)^{\alpha-1}(k-k^*) - (\delta+n)(k-k^*)$$

Note: $(k^*)^{\alpha-1} = \frac{\delta+n}{s}$

$$\Rightarrow \boxed{\dot{k} = (\delta+n)(\alpha-1)(k-k^*)}$$

$\alpha = 1/3 \quad \delta = .05 \quad n = .01$

$\Rightarrow (\delta+n)(\alpha-1) \approx .04 - .05$

Observed: $.02$ (Barro + Sala-i-Martin (1992)).

4) Normative Analysis Not Possible

- How fast should an economy grow?

Cass-Koopmans Model

Assumptions

1.) Representative ∞-horizon agent/dynastic family
   - complete mkts.

2.) Perfect Comp./Constant Returns

3.) Exogenous Labor + Technology

Planner's Problem

- 2nd Welfare Theorem (P.O $\Rightarrow$ Comp. Equil.)
- Later we'll discuss market decentralization

$$\max_{c} \int_0^\infty u(c)e^{-(\rho-n)t}\,dt$$

$$\text{s.t.} \quad \dot{k} = f(k) - (\delta+n)k - c$$

Current-Valued Hamiltonian

$$H^{cv} = u(c) + \lambda[f(k) - (\delta+n)k - c]$$

FOCs

1.) $H_c = 0$: $\quad u'(c) - \lambda = 0$

2.) $\dot{k} = H_\lambda$: $\quad \dot{k} = f(k) - (\delta+n)k - c$

3.) $\dot{\lambda} = (\rho-n)\lambda - H_k$: $\quad \dot{\lambda} = (\rho-n)\lambda - \lambda[f'(k) - (\delta+n)]$

From 1. $\quad \dot{\lambda} = u''(c)\,\dot{c}$

Sub into 3.

$$u''(c)\,\dot{c} = u'(c)[\rho + \delta - f'(k)]$$

Or

$$\frac{\dot{c}}{c} = \frac{-u'(c)}{cu''(c)}[f'(k) - (\rho+\delta)]$$

Assume $u = \dfrac{c^{1-\theta}}{1-\theta} \Rightarrow \dfrac{-u'}{cu''} = \dfrac{1}{\theta}$

θ = Constant Relative Risk Aversion

$\frac{1}{\theta}$ = Intertemporal Marginal Rate of Substitution

- Hence, the optimal path is characterized by the following first-order (nonlinear) system

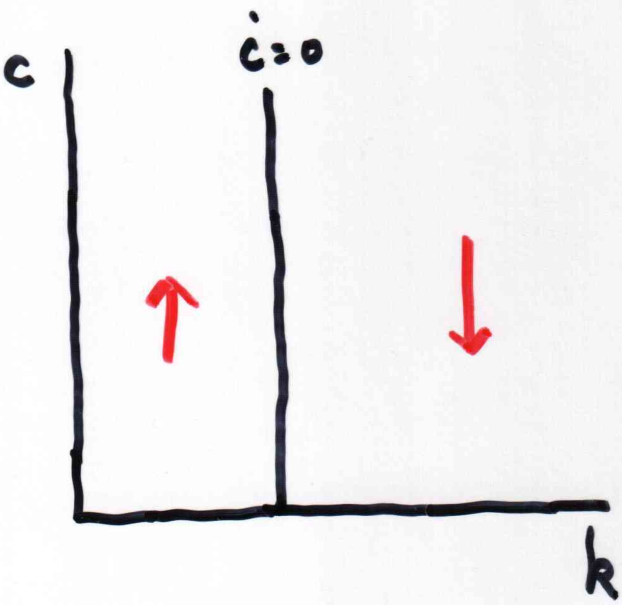
$$\left.\boxed{\begin{aligned} \dot{k} &= f(k) - (\delta+n)k - c \\ \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[f'(k) - (\rho+\delta)\right] \end{aligned}}\right\} \text{Cass-Koopmans}$$

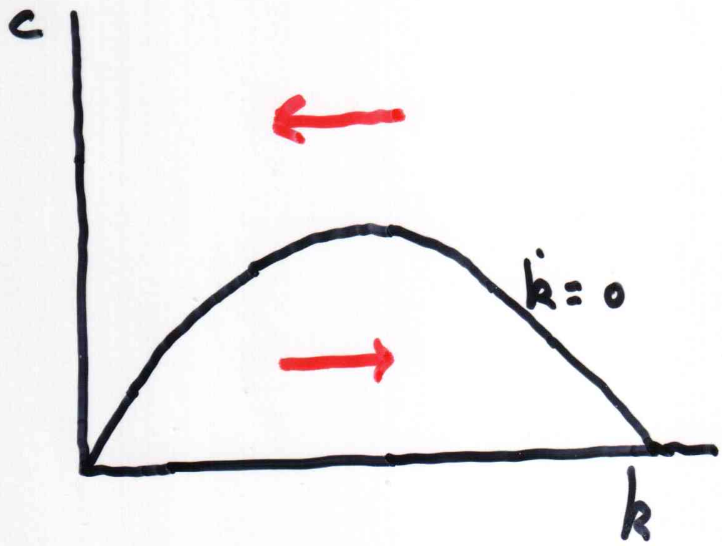
Along with the Transversality Condition,

$$\boxed{\lim_{t\to\infty} e^{-(\rho-n)t}\lambda(t)k(t) = 0}$$

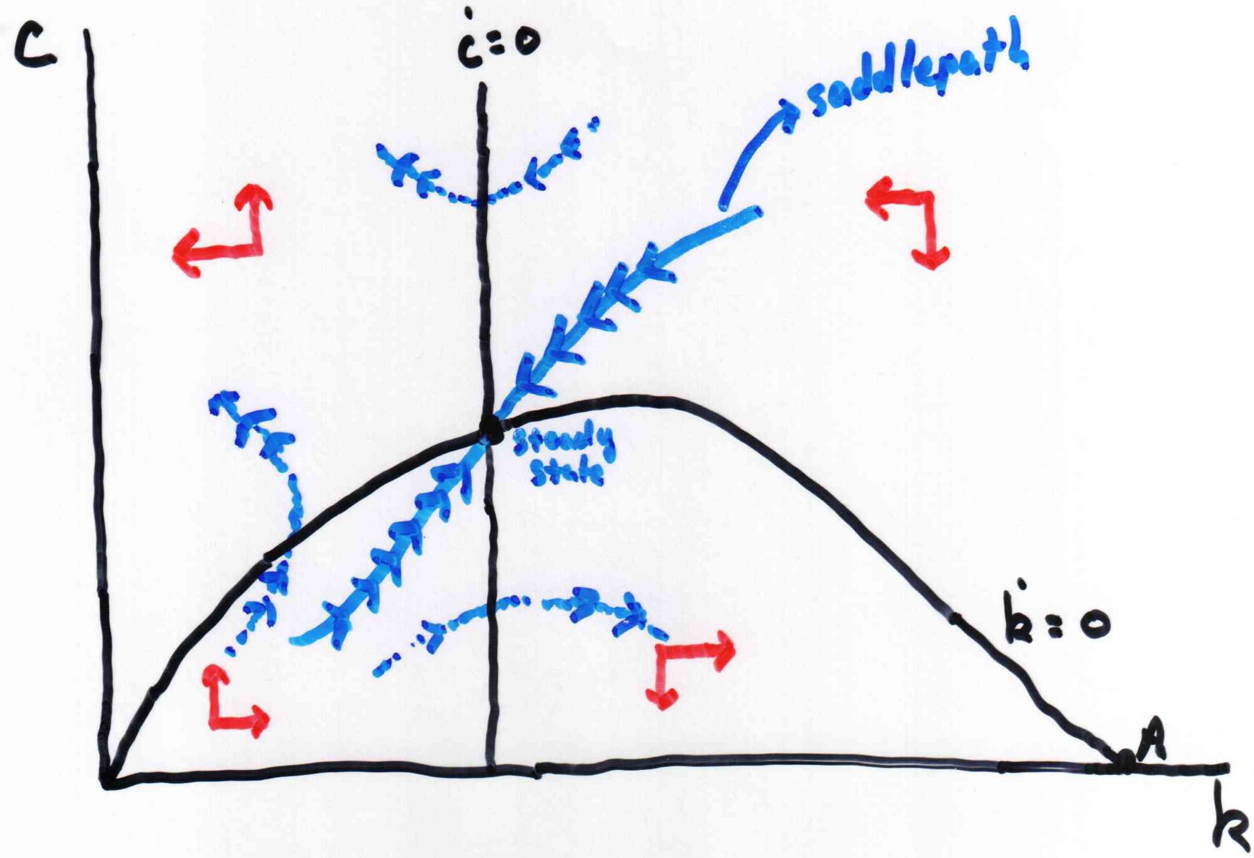
Phase Diagram

- We can analyze this system using a "phase diagram"





• Putting them together,



• All paths except the saddlepath either hit the vertical axis and violate the Euler eq. (c jumps to 0 when $k=0$), or converge to pt. A, which violates the TVC (because $u'(c)=\lambda$ rises faster than $\rho-n$).

Comments

1.) Given an initial $k(0)$, $c(0)$ must jump up to the saddlepath. Along the saddlepath c & k move together until they converge to the steady state (i.e., $c = g(k)$, where $g(\cdot)$ is like a DP policy function).

2.) At the steady state, Y, K, and C all grow at rate n (balanced growth). Per capita output & consumption converge to constants unless there is (exogenous) tech. progress. Hence, like Solow, the CK model fails to explain (sustained) growth in living standards.

3.) At the steady state, consumption is below the "Golden Rule"

4.) Depending on parameters, convergence rate to the steady state can be either slower or faster than Solow.

Steady State Saving Rate

- At steady state,

$$f(k^*) - (\delta + n)k^* - c^* = 0 \qquad s^* = 1 - \frac{c^*}{f(k^*)}$$

$$\Rightarrow \frac{c^*}{f(k^*)} = 1 - (\delta + n)\frac{k^*}{f(k^*)}$$

With Cobb-Douglas,

$$f' \cdot k = \alpha f \implies \frac{k^*}{f(k^*)} = \frac{\alpha}{f'(k^*)}$$

From s.s. Euler,

$$f'(k^*) = \rho + \delta$$

Therefore,

$$\boxed{s^* = (\delta + n)\frac{k^*}{f(k^*)} = (\delta+n)\frac{\alpha}{f'(k^*)} = \frac{(\delta+n)\alpha}{\delta+\rho}} < \alpha$$

Golden Rule saving rate

Example

$\delta = .05$
$\rho = .02$
$n = .01$
$\alpha = 1/3$

$\implies s^* = 28.6\%$ (gross saving rate).

- What about θ? Influence s^* if $g = \dot{A}/A \neq 0$

$$s^* = \frac{\alpha(\delta + n + g)}{\delta + \rho + \theta g}$$

- However, main influence of θ is during transition
- Suppose $k(0) < k^*$. Two offsetting effects on saving:

1.) Income below s.s.
$\Rightarrow$ want to borrow to smooth consumption } Income Effect

2.) Low $k \Rightarrow$ high MPK/high interest rate
$\Rightarrow$ return on saving is high } Substitution Effect

High $\theta \Rightarrow$ Low intertemp. substitution
$\Rightarrow$ Income effect > Substitution effect
$\Rightarrow$ Saving rate lower than s^*
$\Rightarrow$ Slow convergence

Low $\theta \Rightarrow$ High intertemp. substitution
$\Rightarrow$ Income effect < Substitution effect
$\Rightarrow$ saving rate higher than at s^*
$\Rightarrow$ Fast convergence

Threshold Values

$\frac{1}{\theta} = s^* \implies$ constant saving rate

empirically realistic case < $\frac{1}{\theta} > s^* \implies$ higher saving rate when $k(0) < k^*$

$\frac{1}{\theta} < s^* \implies$ lower saving rate when $k(0) < k^*$

Local Approximation (around s.s.) to Convergence Rate

$$\dot{k} \approx \frac{\partial \dot{k}}{\partial k} \cdot (k - k^*) + \frac{\partial \dot{k}}{\partial c} (c - c^*)$$

$$\dot{c} \approx \frac{\partial \dot{c}}{\partial k} \cdot (k - k^*) + \frac{\partial \dot{c}}{\partial c} \cdot (c - c^*)$$

$$\frac{\partial \dot{k}}{\partial k} = f'(k^*) - (n+\delta) = \rho + \delta - (n+\delta) = \rho - n$$

$$\frac{\partial \dot{k}}{\partial c} = -1$$

$$\frac{\partial \dot{c}}{\partial k} = \frac{c^*}{\theta} f''(k^*)$$

$$\frac{\partial \dot{c}}{\partial c} = \frac{1}{\theta}\left[f'(k^*) - (\rho+\delta)\right] = 0$$

- Write in matrix form,

$$\boxed{\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{bmatrix} \rho - n & -1 \\ \frac{c^*}{\theta} f''(k^*) & 0 \end{bmatrix} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}}$$

$\det = \frac{c^* f''}{\theta} < 0$

$\Rightarrow$ eigenvalues have opposite signs

$\Rightarrow$ saddlepath stability

Eigenvalues

$$\begin{vmatrix} \rho - n - \lambda & -1 \\ \frac{c^*}{\theta} f''(k^*) & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (\rho - n)\lambda + \frac{c^* f''}{\theta} = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{\rho - n \pm \sqrt{(\rho-n)^2 - 4\frac{c^* f''}{\theta}}}{2}$$

Let $\lambda_1 < 0$, $\lambda_2 > 0$ $\quad \lambda_1 = \frac{1}{2}\left[(\rho - n) - \sqrt{(\rho-n)^2 - 4\frac{c^* f''}{\theta}}\right]$

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

eigenvector associated with stable eigenvalue (saddlepath)

eigenvector associated with unstable eigenvalue $= 0$ by TVC.

- Stable eigenvector / Saddlepath

$$\begin{bmatrix} \rho - n - \lambda_1 & -1 \\ \frac{c^*}{\theta} f''(k^*) & -\lambda_1 \end{bmatrix} \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = 0$$

From initial condition, $c_{11} = k(0) - k^*$

jump to saddlepath

$$\Rightarrow c_{21} = (\rho - n - \lambda_1)(k(0) - k^*) = c(0) - c^*$$

$$\Rightarrow k(t) = k^* + [k(0) - k^*] e^{\lambda_1 t}$$
$$c(t) = c^* + [c(0) - c^*] e^{\lambda_1 t}$$

Local Approx. to optimal time paths

Convergence Rates

Sub in for k^*, c^*

$$\lambda_1 = \frac{1}{2}\left[(\rho - n) - \sqrt{(\rho - n)^2 + \frac{4}{\theta} \frac{1-\alpha}{\alpha} (\rho + \delta)[\rho + \delta - \alpha(\delta + n)]} \right]$$

Remember Solow: $\lambda_1 = -(1-\alpha)(n+\delta)$

$\rho = .02$
$n = .01$
$\theta = 1$
$\alpha = 1/3$
$\delta = .05$

$\Rightarrow \lambda_1 = -.079$

Solow $= -.04$

However, if $\theta = 8$ then $\lambda_1 = -.025$