Topics for Today

1.) The Cass-Koopmans Model

- The Planner's Problem
- Phase Diagram + Saddlepath
- Linearization / Local Convergence Rates

from last time

- Comparative Dynamics
- Market Decentralization

2.) The Diamond Model

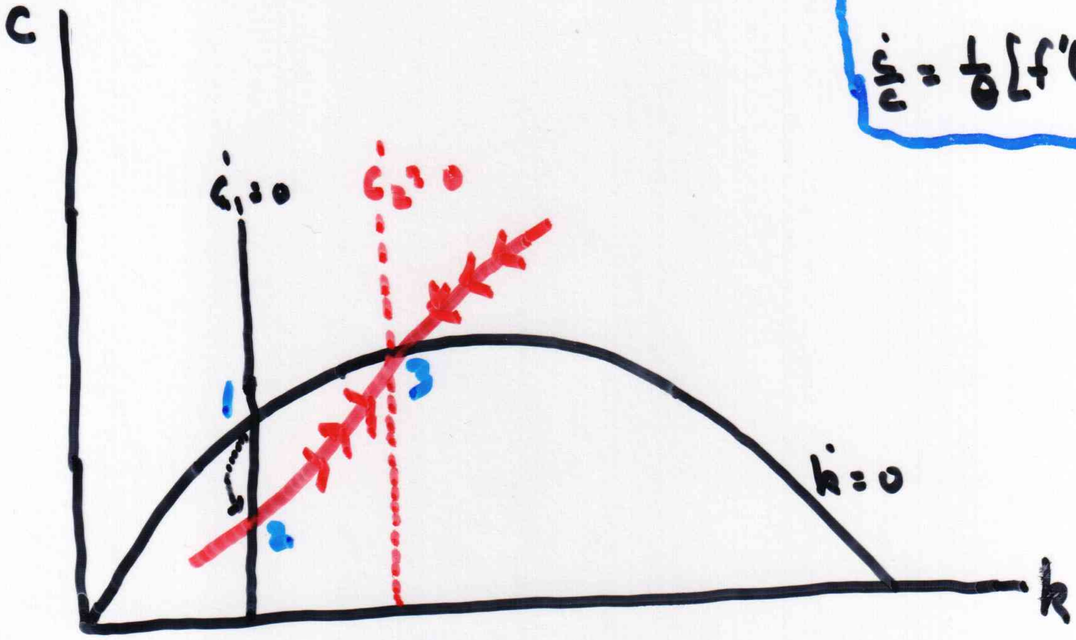
- Dynamic Inefficiency

Comparative Dynamics

① $\rho \downarrow$ (unanticipated, permanent)

- This is the analog of an increase in the saving rate that is often considered in the Solow model.

$$\boxed{\begin{aligned} \dot{k} &= f(k) - (\delta+n)k - c \\ \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[f'(k) - \rho + \delta\right] \end{aligned}}$$



- Pt. 1 = Initial Steady State
  Pt. 2 = Immediate effect
  Pt. 3 = New Steady State

- Consumption initially falls. Over time, capital and consumption increase. Eventually, per capita output will be higher.

- However, as in Solow, there is only a <u>temporary</u> increase in growth rate.

② $G\uparrow$ (unanticipated, permanent)

- Let's incorporate the govt. in the simplest possible way:

1.) Govt. purchases are separable in utility [don't affect $u'(c)$]

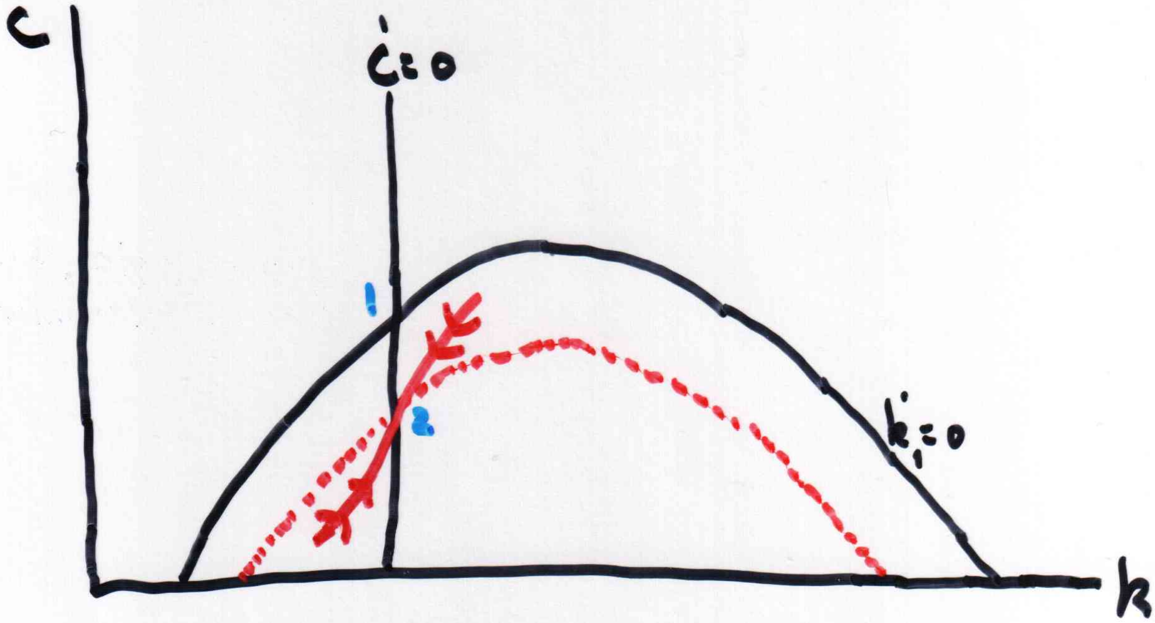
2.) Govt. purchases are not productive [don't affect $f(k)$].

3.) Taxes are lump sum [Ricardian Equivalence].

- Later we'll discuss the effects of distortionary taxes in the Cass-Koopmans model.

Letting g = per capita govt. purchases, we now have:

$$\boxed{\dot{k} = f(k) - (\delta+n)k - c - g \qquad \frac{\dot{c}}{c} = \frac{1}{\theta}[f'(k) - (\rho+\delta)]}$$



- Consumption immediately jumps to the new steady state at Pt. 2. Interest rate, investment, output, growth rate, etc., <u>not</u> affected. What if the change was temporary?

Market Decentralization

- With complete markets, there are many ways to decentralize the Pareto allocation. Let's suppose firms rent their capital from households, while households own the firms.

- Let a = per capita assets/wealth. (In equilibrium, the only asset that is held is claims to capital, so $a = k$).

- Now the household's problem is:

$$\max_{c} \int_0^\infty u(c)\, e^{-(\rho - n)t}\, dt$$

s.t.

$$c + \dot{a} = w + ra - na \quad \Big\} \text{ per capita budget constraint}$$

Euler Eq.: $\dfrac{\dot{c}}{c} = \dfrac{1}{\theta}[r - \rho]$

Firm's Problem

$$\max_{K,L} \; F(K,L) - \overset{\text{rental rate}}{R} \cdot K - w \cdot L$$

FOCs

$$F_K = R = r + \delta = f'(k)$$

$$F_L = w = f(k) - k \cdot f'(k)$$

From Euler's Theorem,

$$w + rk = f(k) - kf'(k) + (f'(k) - \delta)k$$

$$= f(k) - \delta k$$

with CRS, factor payments = net national product

Substituting into household's budget constraint, and imposing equil. condition $a = k$ yields

$$c + \dot{k} = f(k) - (n+\delta)$$

same as Pareto problem

Finally, sub $r = f'(k) - \delta$ into household Euler eq.

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'(k) - (\rho + \delta)\right]$$

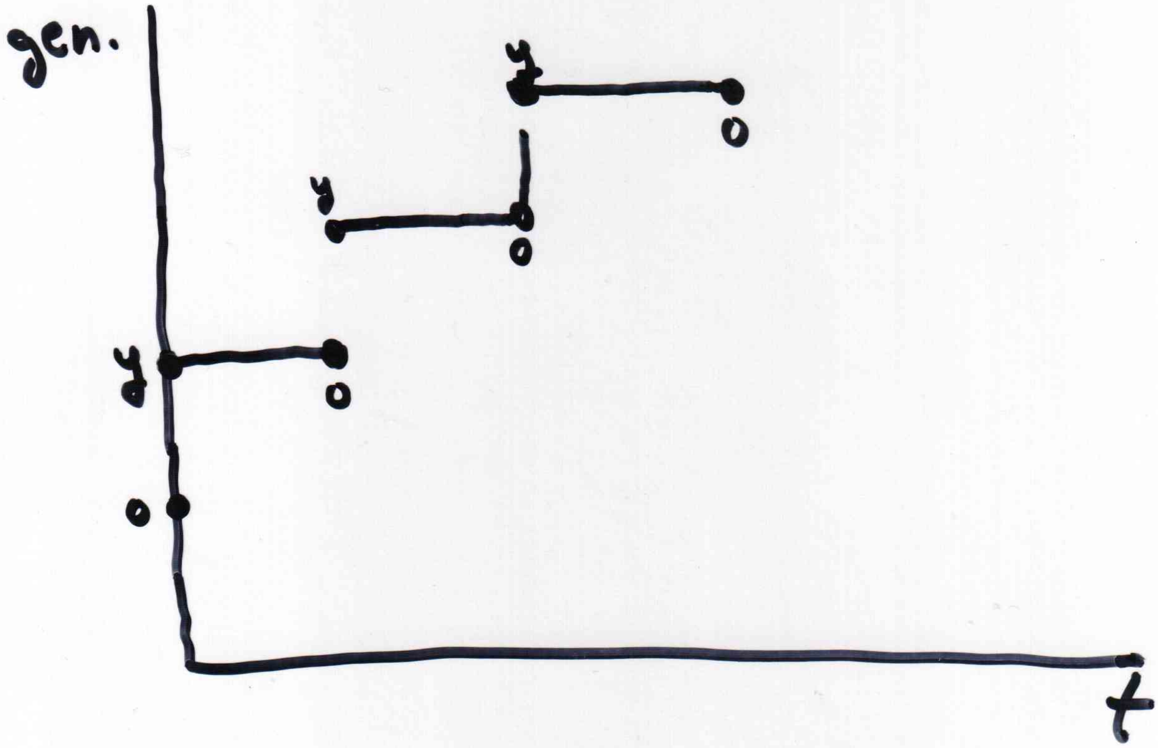
same as Pareto problem.

Diamond Model

- Diamond model incorporates production into the original Samuelson (1958) OLG model. This model is widely used to study public finance issues, especially when demographic or intergenerational issues are important, e.g., social security reform.

Assumptions

1.) Infinite horizon economy consisting of 2-period lived overlapping generations



2.) Individuals do not care about posterity.

3.) Agents are endowed with 1 unit of labor when "young", and 0 units when "old". Hence, agents must provide for their old age by saving some of their 1st period labor income.

4.) Competitive firms produce output by employing young workers and renting capital from old people. [Note: factors are supplied inelastically].

5.) Population grows at rate n, i.e., $L_{t+1} = (1+n)L_t$

New Issues

1.) Multiple Equilibria + Sunspots. (In contrast, standard preference + tech. assumptions deliver unique steady states in Solow + Cass-Koopmans).

2.) Dynamic Inefficiency. The competitive equil. may not be Pareto Optimal, despite perfect competition + no externalities. This does not reflect "incomplete markets", arising from the fact that you can't trade with the "unborn". Even with time-0 AD mkts., the CE may not be PO! What happened to the 1st Welfare Theorem?

Households

Let c_{1t} = consumption of young at time-t

c_{2t} = consumption of old at time-t

The household's problem is:

$$\max_{c_{1t},\, c_{2t+1}} \quad \frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_{2t+1}^{1-\theta}}{1-\theta}$$

$$\text{s.t.} \quad c_{1t} + \overset{\text{saving}}{S_t} = \overset{\text{wage income}}{W_t}$$

$$c_{2t+1} = (1+r_{t+1}) S_t$$

Euler Eq.

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho} \right)^{1/\theta}$$

$$\Longrightarrow c_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta}} c_{1t} = W_t$$

$$\Longrightarrow c_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} W_t$$

Therefore,

$$S_t = s(r_{t+1})\,W_t$$

where

$$s(r) = \frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta} + (1+r)^{\frac{1-\theta}{\theta}}}$$

Note, in general, $s'(r)$ could be either positive or negative [i.e., offsetting income & substitution effects]. With these CRRA prefs. we have the following results:

a.) $s'(r) > 0$ if $\theta < 1$

b.) $s'(r) < 0$ if $\theta > 1$

c.) If $\theta = 1 \Rightarrow$ log utility
$\Rightarrow$ saving is indpt. of interest rate.

Firms

Firms & factor mkts. are competitive, so wage & rental rate are given by marginal products:

$$W_t = f(k_t) - k_t f'(k_t)$$

$$r_{t+1} = f'(k_{t+1}) - \delta$$

Market Equilibrium

- In equilibrium, the quantity of capital demanded by firms must equal the quantity supplied by old people, which is equal to last period's aggregate saving,

$$K_{t+1} = S_t \cdot L_t$$

$$\Rightarrow \quad (1+n)k_{t+1} = s(r_{t+1})\, W_t$$
$$= s(f'(k_{t+1}) - \delta)\left[f(k_t) - k_t \cdot f'(k_t)\right]$$
$$= H(k_{t+1}, k_t)$$

Steady State

- A s.s. satisfies $\bar{k} = \frac{1}{1+n} H(\bar{k}, \bar{k})$. In general, this could have multiple solutions.

- A s.s. will be stable iff

$$\left| \frac{dk_{t+1}}{dk_t} \right| = \left| \frac{\frac{1}{1+n} H_2}{1 - \frac{1}{1+n} H_1} \right| < 1$$

Special Case

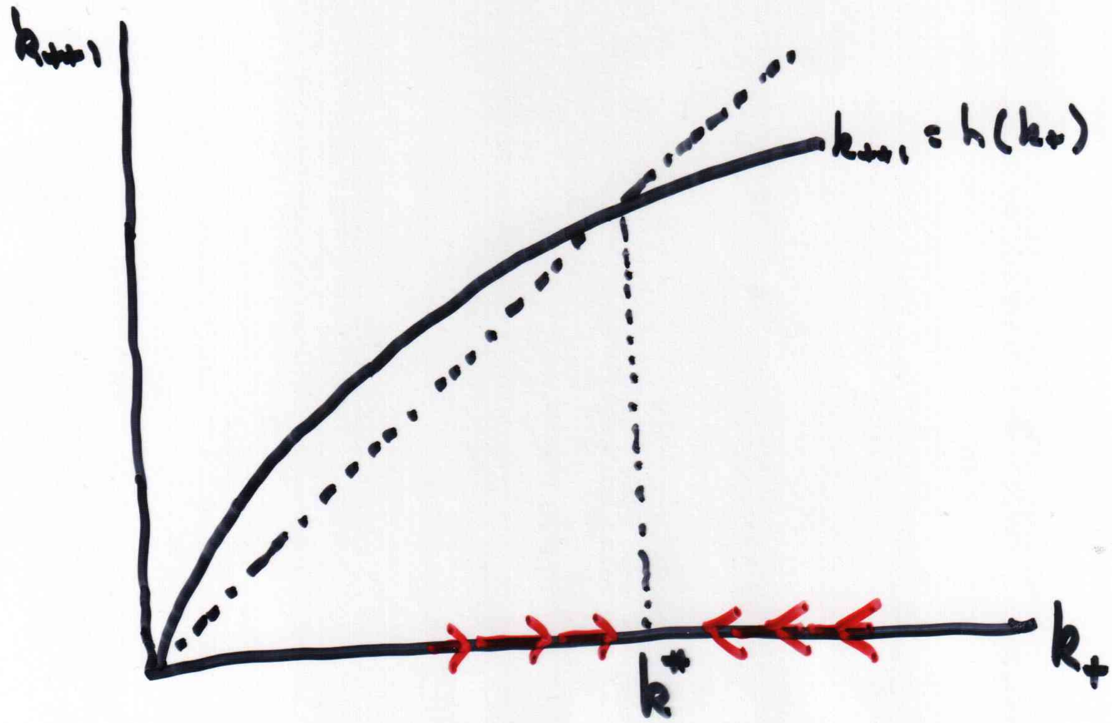
Suppose log utility ($\theta = 1$) and Cobb-Douglas production, $f(k) = k^\alpha$. Then we have

$$\boxed{k_{t+1} = \frac{1-\alpha}{(1+n)(2+\rho)} k_t^\alpha}$$

$$\Longrightarrow \quad k^* = \left[\frac{1-\alpha}{(1+n)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

$$\Longrightarrow \quad r^* = f'(k^*) - \delta$$

$$= \frac{\alpha(1+n)(2+\rho)}{1-\alpha} - \delta$$



Comments

1.) As with Solow + Cass-Koopmans, Diamond does not generate <u>sustained</u> per capita growth without exogenous tech. progress.

2.) Interestingly, the market equilibrium may be <u>inefficient</u> (Pareto improvable). If there is a strong desire to save, it may be the case that $r^* < n$. In this case, the market interest rate is lower than the technological/biologically feasible rate of interest. If a planner can effect a direct transfer from the young to the <u>current</u> old, each generation gets a rate of return of n, which makes <u>everyone</u> better off.

<u>Examples ?</u> : 1.) Pay-as-you-go Social Security
2.) Govt. debt
3.) Fiat money

- Note: This inefficiency does not reflect externalities, or incomplete mkts., more generally. The 1st Welfare Theorem breaks down because when $r < n$ the value of the aggregate endowment becomes infinite.

Calibration

- Suppose we think of a "period" as 30 years

Pop. growth = 1% per annum $\Rightarrow n = (1.01)^{30} - 1 = .348$

Capt. Depr. = 5% per annum $\Rightarrow \delta = 1 - (.95)^{30} = .785$

Time Pref. = 3% " " $\Rightarrow \rho = (1.03)^{30} - 1 = 1.427$

$\alpha = 1/4$

Then $\boxed{r^* = .754 > .348 = n \Rightarrow \text{dynamic efficiency}}$

However, suppose agents are more patient,

let Time Pref. = 1% $\Rightarrow \rho = (1.01)^{30} - 1 = .348$

Then $\boxed{r^* = .270 < .348 = n \Rightarrow \text{dynamic inefficiency}}$

- Note, to apply this to the real world, must recognize that asset returns are risky, and therefore embody risk premia. Which asset return should we use for r?