Topics for Today

1.) Endogenous Growth

- AK models
- Externalities
- Human Capital
- R+D

Endogenous Growth

- The Solow, Cass-Koopmans, + Diamond models are useful for analyzing a variety of issues (e.g., convergence + transition dynamics). However, they do not generate sustained growth in per capita incomes. At best, they merely describe the growth process, rather than explain it.
- The basic reason why these models do not generate sustained growth is the presence of diminishing returns to capital. All models of endogenous growth escape the forces of diminishing returns in one way or another.

Alternative Approaches to Endogenous Growth

1.) All factors reproducible
   - human capital (Lucas 1988)

2.) Relax Inada Condition ($f'(k) \neq 0$ as $k \to \infty$)
   - CES vs. Cobb-Douglas

3.) External Economies of Scale
   - Knowledge Spillovers (Romer 1986)

4.) Internal Economies of Scale
   - R+D (Romer 1990)

- The first 3 are analytically straightforward, since they are consistent with competitive markets. R&D models are complicated, as they require general equilibrium models of imperfect competition.

- The first 2 approaches generally yield Pareto optimal growth rates. The second 2 approaches generally yield Pareto suboptimal growth. This raises the possibility of Pareto improving government policies.

AK Models

- Suppose we think of labor as just another type of capital, which can be accumulated by investing in education/human capital

- Let's maintain the 1-sector production technology of Solow/Cass-Koopmans, so that human capital is accumulated by foregoing consumption (and is reversible). Later we'll discuss a 2-sector human capital model (i.e., Lucas (1988)).

Production Function

$Y = K^{1-\alpha} H^{\alpha}$ (H = "human capital")

(Note: CRS production).

We can write this as:

$Y = K \cdot f(H/K)$, where $f = (H/K)^{\alpha}$

- In equilibrium, both factors are paid their marginal products:

$$MPK = f - (H/K)\cdot f' = R_K = r + \delta_K$$
$$MPH = f' = R_H = r + \delta_H$$

- In equilibrium, the net rate of return to each capital input must be equal to the interest rate. This yields the no arbitrage condition:

$$\boxed{f - H/K\, f' - \delta_K = f' - \delta_H}$$ } Equal net rates of return

$\Longrightarrow$ unique value of $h^* = (H/K)^*$

• Now, if we define $A = f(h^*)$, we have

$$\boxed{Y = A \cdot K} \quad \Big\} \text{ AK Production Function}$$

Equilibrium

• In equilibrium, $r = A - \delta$, so we have:

$$\boxed{\begin{aligned} \dot{k} &= (A - \delta - n)k - c \\ \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[A - \delta - \rho\right] \\ \lim_{t\to\infty} &\left[k(t)\, e^{-(A-\delta-n)t}\right] = 0 \end{aligned}}$$

• This has the following closed-form solution:

$$c(t) = c(0)\, e^{\frac{1}{\theta}(A-\delta-\rho)t}$$

$$k(t) = \frac{c(0)}{\phi}\, e^{\frac{1}{\theta}(A-\delta-\rho)t}$$

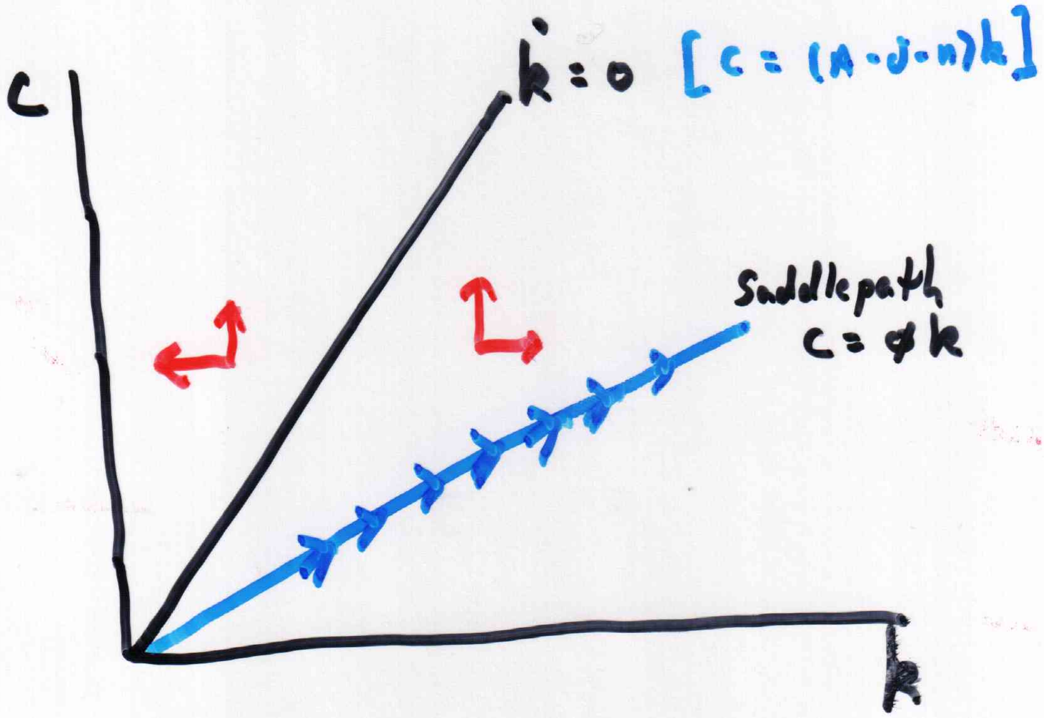
where $\phi = \left(\frac{\theta-1}{\theta}\right)(A-\delta) + \rho/\theta - n$

• Note, endogenous growth occurs iff

$$\boxed{A > \rho + \delta > (A-\delta)\cdot(1-\theta) + \theta n + \delta}$$

(The second $\neq$ ensures the TVC is satisfied).

Phase Diagram



CES Production / Relaxation of Inada Conditions

- The AK model features no transition dynamics. Growth is always constant. This is inconsistent with observed (conditional) convergence.

- We can obtain results similar to AK model, while generating transition dynamics, if we generalize the production function from Cobb-Douglas to CES.

- Now suppose

$$\boxed{Y = F(K,L) = B \cdot \left[a \cdot (bK)^{\psi} + (1-a)\left[(1-b)L\right]^{\psi} \right]^{1/\psi}}$$

where $\frac{1}{1-\psi}$ is the elasticity of substitution between K and L [Cobb-Douglas is the special case when $\psi = 0$].

- We will see that if $\psi < 1$, so that K & L are relatively substitutable, then the model ~~will~~ can generate endogenous growth.

- As with Cobb-Douglas, this CES production function features CRS, and can be written in per capita terms as follows:

$$y = f(k) = B \cdot \left[a \cdot (bk)^{\psi} + (1-a)(1-b)^{\psi}\right]^{1/\psi}$$

- The crucial thing to note here is that:

$$\boxed{\lim_{k \to \infty} [f'(k)] = B b a^{1/\psi} > 0}$$

which violates the usual Inada condition

$$\lim_{k \to \infty} f'(k) = 0$$

- If we define $A = B b a^{1/\psi}$ then, like the AK model, this model generates endogenous growth if $A > \delta + \rho$. [Note: this is more likely to be the case for high elasticity of substitution].

Exercise: Construct the phase diagram. (Hint: define the transformed variables $z = \frac{f(k)}{k}$ and $x = c/k$).

- Note that if we define $\Omega(K,L) = F(K,L) - AK$, where $F(\cdot)$ is the above CES production function, then $\Omega(\cdot)$ satisfies the usual Inada conditions and we can write:

$$Y = A \cdot K + \Omega(K,L)$$

The $A \cdot K$ component generates sustained growth, while the $\Omega(K,L)$ component generates the usual transition dynamics.

Question: We originally specified a Cobb-Douglas form because it is consistent with the observation that factor shares are (approximately) constant over time. Is this true with the above CES function? What happens to labor's share as $t \to \infty$?

Externalities / Knowledge Spillovers

- Assume production takes the usual form

$$Y_i = K_i^{\alpha} (A_i L_i)^{1-\alpha} \quad \} \text{ output of firm } i$$

- Romer (1986) hypothesized

$$A_i = B \bar{K}$$

where $\bar{K}$ = aggregate capital stock. This involves 2 key assumptions:

1.) Knowledge creation is a by-product of investment. Since it depends on the stock of capital (history of investment) it is sometimes called "learning-by-doing".

2.) The effect is external to the individual firm. Each firm takes A as given when making its own investment decisions.

- Substituting in for A, and imposing the equilibrium condition $K_i = \bar{K}$ we get

$$Y = B^{1-\alpha} L^{1-\alpha} \bar{K}$$
$$= A \cdot \bar{K}$$ (assuming L is constant)

- Hence, we get an $A \cdot K$ model, which generates endogenous growth as before.

- The key difference here is that the equilibrium is not Pareto Optimal (due to the externality).

Problems

1.) As with AK model, this model lacks transition dynamics, and is inconsistent with observed (conditional) convergence.

2.) The model features "scale effects". Note that the growth rate depends positively on L (the population/labor force). There is little evidence for this in the data (though see Kremer (QJE, 1993)).

The Lucas (1988) 2-sector Human Capital Model

- Now assume economy consists of **two** sectors:

1.) Goods producing sector

2.) Education/Human Capital producing sector

Key Assumption: Human Capital accumulation is **not** subject to diminishing returns (society vs. individual).

Goods Production: $Y = AK^{\alpha}(uH)^{1-\alpha} = C + \dot{K} + \delta K$

Human Capital Accumulation: $\dot{H} = B(1-u)\cdot H - \delta H$

Note: 1.) u = fraction of time spent in goods producing sector

2.) For simplicity, K & H assumed to depreciate at same rate

Preferences : $\max_c \int_0^\infty u(c) e^{-\rho t} dt \qquad u(c) = \frac{c^{1-\theta}}{1-\theta}$

Note : It is clear that the steady state growth rate of $H, K, Y, + C$ is determined by $(1-u)$ = fraction devoted to schooling.

- u is the outcome of a dynamic optimization problem with 2 control variables (c, u) and 2 state variables (K, H).

Present-Value Hamiltonian

$$\boxed{\mathcal{H} = u(c)e^{-\rho t} + v\left[AK^\alpha (uH)^{1-\alpha} - c - \delta K\right] + \mu\left[B(1-u)H - \delta H\right]}$$

- We can simplify the analysis by defining the following change-of-variables:

$$\boxed{\begin{aligned} \omega &= K/H \\ \chi &= c/K \end{aligned}}$$

- With these change-of-variables the resource constraints become

$$\dot{K}/K = Au^{1-\alpha}w^{-(1-\alpha)} - \chi - \delta$$

$$\dot{H}/H = B\cdot(1-u) - \delta$$

Therefore,

$$\boxed{\frac{\dot{w}}{w} = \dot{K}/K - \dot{H}/H = Au^{1-\alpha}w^{-(1-\alpha)} - \chi - B\cdot(1-u)}$$

- This will be one of the 3 key eqs. of our model. To derive the other 2 we need to consider the Hamiltonian:

$\frac{\partial \mathcal{H}}{\partial c} = 0$ ①: $u'(c) = \nu e^{\rho t}$

$\frac{\partial \mathcal{H}}{\partial u} = 0$ ②: $\frac{\mu}{\nu} = (A/B)(1-\alpha)u^{-\alpha}w^{\alpha}$

$\dot{\nu} = -\frac{\partial \mathcal{H}}{\partial K}$ ③: $\frac{\dot{\nu}}{\nu} = -A\alpha u^{1-\alpha}w^{-(1-\alpha)} + \delta$

$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial H}$ ④: $\frac{\dot{\mu}}{\mu} = -(\nu/\mu)A(1-\alpha)u^{1-\alpha}w^{\alpha} - B(1-u) + \delta$

- Differentiate (1) w.r.t. time and use (3)

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[A\alpha u^{1-\alpha} w^{-(1-\alpha)} - \delta - \rho\right]$$

Therefore,

$$\boxed{\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \dot{k}/k = \left(\frac{\alpha-\theta}{\theta}\right)\cdot A u^{1-\alpha} w^{-(1-\alpha)} + x - \frac{1}{\theta}\left[\delta(1-\theta)+\rho\right]}$$

This is the 2nd key equation.

- Finally, differentiate (2) w.r.t. time and use (3) and (4)

$$\boxed{\frac{\dot{u}}{u} = B\cdot\frac{1-\alpha}{\alpha} + Bu - x}$$

We now have 3 diff. eqs. in (u, x, w).

• Collecting eqs., we have the system:

$$\boxed{\begin{aligned} \frac{\dot{w}}{w} &= A u^{1-\alpha} w^{-(1-\alpha)} - \chi - B\cdot(1-u) \\ \frac{\dot{\chi}}{\chi} &= \left(\frac{\alpha-\theta}{\theta}\right)\cdot A u^{1-\alpha} w^{-(1-\alpha)} + \chi - \frac{1}{\theta}\left[\delta(1-\theta)+\rho\right] \\ \frac{\dot{u}}{u} &= B\cdot\frac{1-\alpha}{\alpha} + B u - \chi \end{aligned}}$$

Steady State

$$w^* = \left(\frac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}}\cdot\left[\phi + \frac{\theta-1}{\theta}\right]$$

$$\chi^* = B\cdot\left[\phi + \frac{1}{\alpha} - \frac{1}{\theta}\right]$$

$$u^* = \phi + \frac{\theta-1}{\theta}$$

where $\phi = \dfrac{\rho + \delta(1-\theta)}{B\theta}$

• Therefore, the economy's steady state per capita growth rate is:

$$g = B\cdot(1-u^*) - \delta$$
$$= B\left(\frac{1}{\theta} - \phi\right) - \delta$$
$$= \boxed{\frac{1}{\theta}(B - \delta - \rho)}$$

> depends on both preferences + technology

Comments

1.) The economy's growth rate is endogenous in the sense that anything that raises the fraction of time devoted to human capital accumulation (e.g., education subsidies) raises the growth rate. However, in this baseline model the competitive equilibrium is Pareto Optimal, and these subsidies would only reduce welfare. [Lucas also studies a version with human capital externalities/spillovers. In that version, education subsidies would be beneficial].

2.) This model features interesting transition dynamics. [Exercise: Derive the phase diagram(s)!] Intuitively, would growth be higher or lower than in the s.s. when $\omega > \omega^*$?

Romer's (1987, 1990) R+D Model

- Although the Lucas model generates sustained growth in the form of higher labor productivity, it does not feature what we think of as technological progress.

- Technological progress has 2 key attributes:
  1.) It involves the development of new goods and new ways of doing things.
  2.) It is often the deliberate outcome of R+D activity

  Romer's model captures these features.

- A key aspect of tech. progress is that it creates goods/ideas that are nonrivalrous. One person's use of the idea does not preclude other people using it. This creates increasing returns that can sustain growth. However, when development of new ideas is a purposeful activity that absorbs resources & requires compensation (i.e., returns are internal to a firm), it precludes the possibility of a competitive equilibrium. Factors cannot be paid their ~~the~~ marginal products, as there would not be enough output to go around!

Assumptions

1.) There are 2 sectors:

a.) A competitive final goods sector

b.) A monopolistically competitive intermediate goods sector

2.) Final output produced according to:

$$\boxed{Y_t = L^{1-\alpha} \int_0^{A_t} Z_t(i)^{\alpha} \, di}$$

where $Z_t(i)$ is intermediate good-i.

3.) $Z_{t+1}(i)$ is produced by one unit of final good output at t.

4.) New intermediate goods can be developed by incurring a one-time fixed cost of k.

5.) The developer of a new intermediate good is granted an infinitely-lived patent on it.

Aggregate Resource Constraint

$$\boxed{C_t + \int_0^{A_{t+1}} Z_{t+1}(i)\,di + (A_{t+1}-A_t)k = L^{1-\alpha}\int_0^{A_t} Z_t(i)^{\alpha}\,di}$$

Since in equilibrium $Z_t(i) = Z_t \;\forall i$, we have

$$C_t + A_{t+1}Z_{t+1} + (A_{t+1}-A_t)k = L^{1-\alpha}A_t Z_t^{\alpha}$$

Comments

1.) Growth due to rising A_t. (Note: New inputs do not erode the MP of existing inputs).

2.) Production function exhibits increasing returns with respect to the three inputs: L, Z and A.

Final Goods Sector

Competition $\Rightarrow$ Factors paid their marginal products

$$w_t = (1-\alpha)L^{-\alpha}\int_0^{A_t} Z_t(i)^{\alpha}\,di$$

1. $$p_t(i) = \alpha L^{1-\alpha} Z_t(i)^{\alpha-1}$$ > price of intermediate good i (derived demand)

Intermediate Goods Sector

$$\pi_t(i) = [P_t(i) - (1+R_m)]\,Z_t(i)$$

> Profit function for intermediate goods firm i

Maxing over $Z_t(i)$ yields

$$P_t(i) = \frac{1+R_m}{1+1/\varepsilon} \qquad \varepsilon = \frac{\partial Z}{\partial P}\cdot \frac{P}{Z} < 0$$

Using the fact that $\varepsilon^{-1} = \alpha - 1$,

2. $$P_t(i) = \frac{1+R_m}{\alpha}$$

> Standard mark-up pricing

Sub 2. into 1.,

3. $$Z_t(i) = \left(\frac{\alpha^2}{1+R_m}\right)^{\frac{1}{1-\alpha}} L$$

Sub 2. and 3. into $\pi_t(i)$ yields maximized profit function

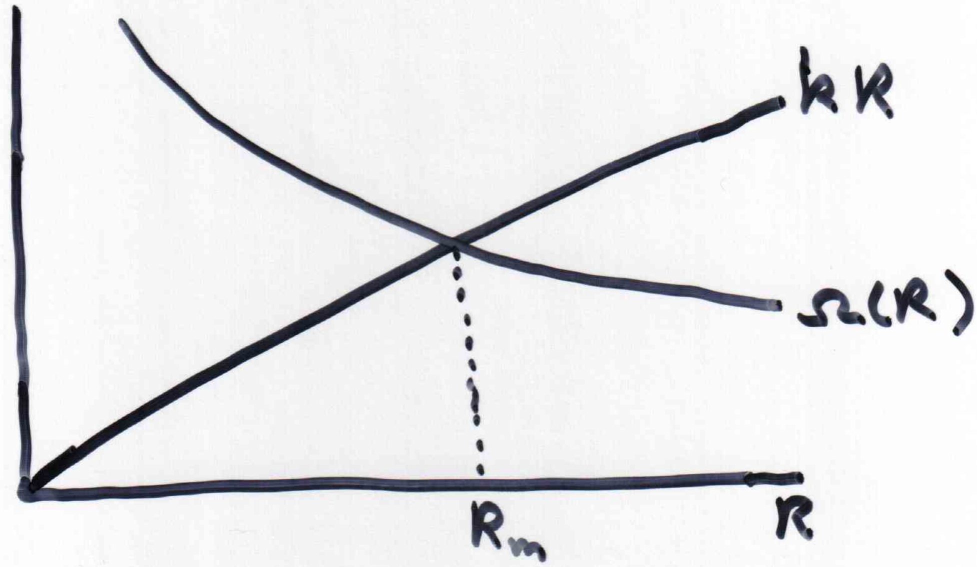
$$\boxed{\pi_t(i) = (1-\alpha)\,\alpha^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{1+R_m}\right)^{\frac{\alpha}{1-\alpha}} L \equiv \Omega(R_m)}$$

Free Entry Equil. Condition (Monop. Comp.).

$$\boxed{\sum_{t=1}^{\infty} (1+R_m)^{-t} \Omega(R_m) = \frac{\Omega(R_m)}{R_m} = k}$$

fixed R&D cost

$$\Rightarrow \Omega(R_m) = k \cdot R_m$$



Euler Eq.

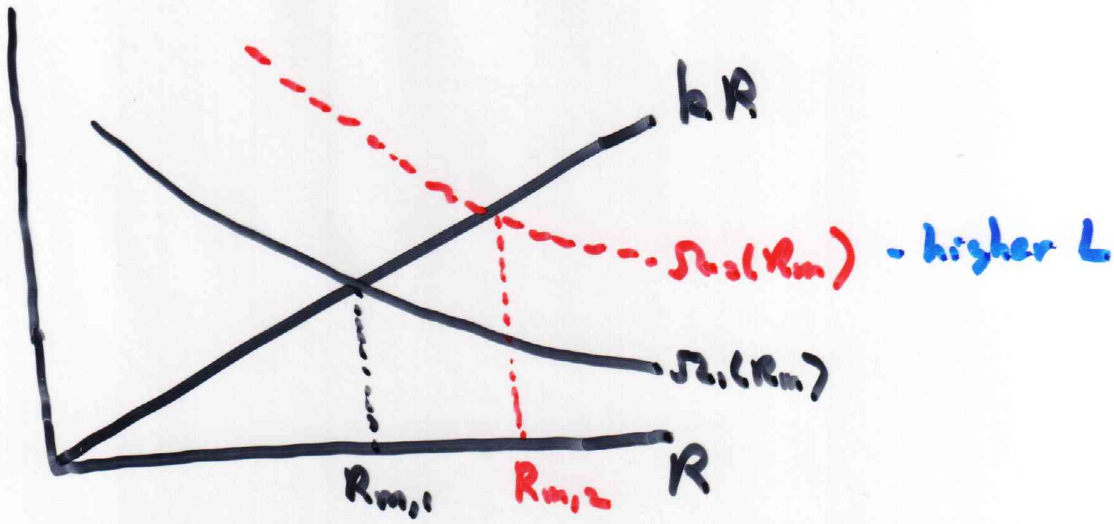
$$\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \beta(1+R_m)$$

$\Rightarrow$ Sustained growth as long as $R_m > \beta^{-1} - 1$

Note: This is more likely when k is lower

Scale Effects

Note what happens when $L \uparrow$



Thus, this model predicts that larger/more populous countries should grow faster. (Intuition: A larger mkt. supports more R+D, since fixed costs can be spread out over a larger sales base).

Optimality

Let R_s = Social Rate of Return

Planner's Problem

$$\max_{z(i)} \; L^{1-\alpha} \int_0^{A_t} z_t(i)^{\alpha} di - (1+R_s) \int_0^{A_t} z_t(i) di$$

FOC w.r.t $z(i)$

(4) $$z_t(i) = \left(\frac{\alpha}{1+R_s}\right)^{\frac{1}{1-\alpha}} L \equiv z_s$$ > constant over time and i

Now max over c_t, A_t

$$\max_{c_t, A_t} \sum \beta^t \frac{c_t^{1-\theta}}{1-\theta}$$

$$\text{s.t. } c_t + A_{t+1} z_{t+1} + (A_{t+1} - A_t) k = L^{1-\alpha} A_t z_t^{\alpha}$$

Euler Eq.

$$-u_t'(\cdot)[z_{t+1} + k] + \beta u_{t+1}'(\cdot)[L^{1-\alpha} z_{t+1}^{\alpha} + k] = 0$$

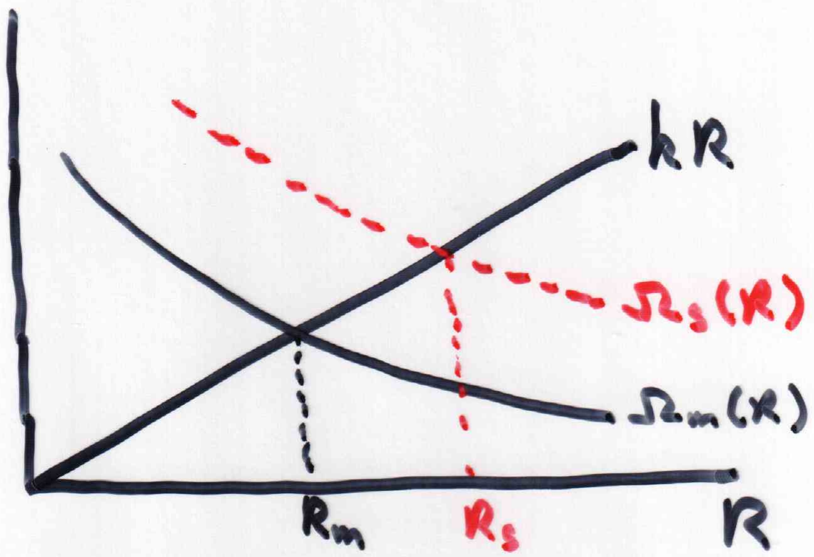
Impose equil. condition $z_{t+1} = z_s$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta \frac{L^{1-\alpha} z_s^{\alpha} + k}{z_s + k} \equiv \beta(1+R_s)$$

Using (4) to sub in for z_s

$$\boxed{k R_s = (1-\alpha)\left(\frac{\alpha}{1+R_s}\right)^{\frac{\alpha}{1-\alpha}} L \equiv \Omega_s(R_s)}$$

- This delivers the following comparison between the market + Pareto optimal outcomes:



__Note__: $\Omega_s(R_s) > \Omega_m(R_m) = \alpha^{\frac{1}{1-\alpha}}\Omega_s(\cdot)$

$\Rightarrow R_s > R_m$

$\Rightarrow$ Optimal Growth > Equil. Growth

__Intuition__: $Z_s > Z_m$ (Monopoly Distortion)