Topics for Today

1.) Review of Arrow-Debreu Asset Pricing
- Equivalent Martingale Measures + "Risk-Neutral Pricing"

2.) The Consumption-Based CAPM (Lucas (1978)).
- When Are Stock Prices Random Walks?

3.) The Equity Premium Puzzle
- Calibrating Risk Aversion

Asset Pricing with Complete Markets

- If markets are complete, AD state prices provide convenient vehicles for many asset pricing problems.
- For simplicity, assume just 2 states & 2 dates (so the distinction between time-0 AD and sequential AD is irrelevant).

Problem

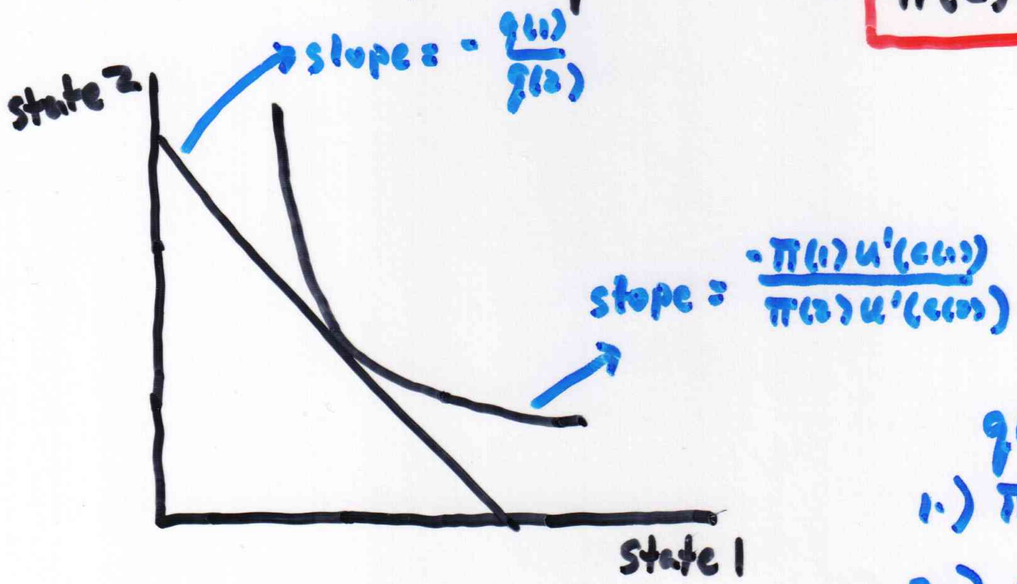
$$\max_{c_0, c(s)} \; u(c_0) + \beta \sum_s \pi(s) u(c(s))$$

$$\text{s.t.} \quad c_0 + \sum q(s) c(s) = y_0 + \sum q(s) y(s)$$

Equil. Conditions

$$u'(c_0) = \lambda$$

$$\beta \pi(s) u'(c(s)) = \lambda q(s) \implies \boxed{\frac{\pi(1) u'(c(1))}{\pi(2) u'(c(2))} = \frac{q(1)}{q(2)}}$$



q(s) will be high when:

1.) $\pi(s)$ is high
2.) $c(s)$ is low

Pricing Redundant Assets

$\boxed{\text{Asset = A bundle of contingent claims}}$

No Arbitrage / "Law of One Price"

$$P(x) = \sum_s q(s) X(s)$$

← payoff in state s

Spanning

Let dim(s) = n

p = vector of n asset prices

X = n×n payoff matrix. Rows are payoffs of each asset across states.

q = n×1 vector of AD state prices

From Law of One Price,

$$P = X \cdot q \implies \boxed{q = X^{-1} \cdot P}$$

if X is full rank

- Hence, with complete mkts., traded assets can be used to construct underlying AD state prices.

Equivalent Martingale Measures / Risk-Neutral Pricing

Write the no arbitrage condition as,

$$P(x) = \sum \pi(s)\left(\frac{q(s)}{\pi(s)}\right)X(s) = E(mX)$$

where

$$m = \frac{q(s)}{\pi(s)} = \beta\,\frac{u'(c(s))}{u'(c_0)} = IMRS$$

stochastic discount factor = $\dfrac{\text{contingent claims price}}{\text{prob. of states}}$

Note,

$$\sum \pi(s)m(s) = E(m) = \sum q(s) = \frac{1}{R_f} = \text{price of risk free claim}$$

Now define,

$\pi^*(s) = R_f\, q(s)$. Note $\sum \pi^*(s) = R^f \sum q(s) = 1$

so π^* is a valid prob. measure. Note that π^* can be written $\pi^*(s) = \dfrac{m(s)\,\pi(s)}{E(m)}$ } Equivalent Martingale Measure

Finally, we have

$$P(x) = \frac{1}{R_f}\sum \pi^*(s)X(s) = \frac{E^*(X)}{R_f}$$

Expectation w.r.t. the equivalent martingale measure

$\boxed{\text{Asset Price: Discounted expected payoff w.r.t. a distorted prob. measure that places higher prob. on states that have high MU of consumption (i.e., low c)}}$

The Lucas '78 Model (The CCAPM)

- Although useful for some purposes, AD asset pricing begs 2 questions:

1.) Where do the AD prices come from? How can they be linked to macro fundamentals? (The previous strategy of inferring q from P is useless when pricing nonredundant assets).

2.) What if markets are incomplete? What can be said about asset prices with incomplete markets?

- Perhaps not surprisingly, the consumption Euler eq. is once again the key starting point.

- Consider an investor who can trade a stock and a bond. He solves:

$$\max \; E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

s.t.

$$c_t + R_t^{-1} L_t + P_t N_t \leq A_t$$

$$A_{t+1} = L_t + (P_{t+1} + y_{t+1}) N_t$$

L_t = bond holdings (claims to 1 unit of consump at $t+1$)

N_t = stock holdings

P_t = price of stock

y_t = dividend

Euler Eqs.

L_t: $$u'(c_t) = \beta R_t E_t u'(c_{t+1})$$

N_t: $$P_t u'(c_t) = \beta E_t\left[u'(c_{t+1})(y_{t+1} + P_{t+1})\right]$$

TVCs

$$\lim_{k\to\infty} E_t \beta^k u'(c_{t+k}) R_{t+k}^{-1} L_{t+k} = 0$$

$$\lim_{k\to\infty} E_t \beta^k u'(c_{t+k}) P_{t+k} N_{t+k} = 0$$

- Note, these eqs. must hold for all investors, regardless of whether mkts. are complete or incomplete. (They might be inequalities for some investors, who face borrowing or short sale constraints)

- As such, they impose testable restrictions on the relationship between asset prices + consumption. These restrictions are the same whether mkts. are complete or incomplete.

- In practice, long time-series of individual consumption data are rarely available. This makes the restrictions difficult to test. However, if mkts. are complete, so that a Rep. Agent exists, then we can test the restrictions on aggregate data.

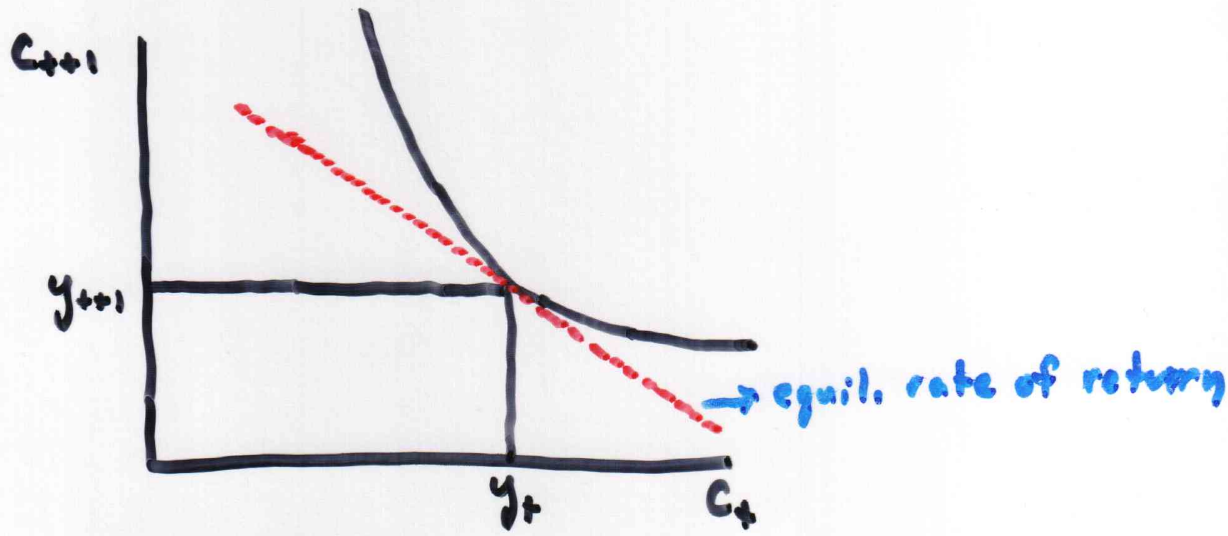
Lucas Model

Complete Mkts. / Endowment Economy

Identical Preferences + Endowments

$\Longrightarrow$ In equil., everyone consumes (per capita) the aggregate endowment, which consists of the stream of dividends from assets ("trees").

Graphically,



Comments

1.) Notice returns adjust so that everyone is content to consume their endowment. In equilibrium, there is no asset trade! These are "shadow prices". (Obviously, $L_t = 0$ in equil., since bonds are in zero net supply).

2.) Notice that Lucas just reverses the logic of Friedman's "Permanent Income Hypothesis". Friedman assumed asset returns were exogenous, and used the consumption Euler eq. to make statements about optimal consumption behavior. (These statements were formally tested by Hall '78). In contrast, Lucas assumes consumption is exogenous, + uses the Euler eq. to make statements about equil. asset prices.

That is,

Friedman/Hall PIH : R_t exogenous / C_t endogenous

Lucas CCAPM Model : R_t endogenous / C_t exogenous

- Of course, both are endogenous. However, the implied restrictions on the empirical relationship between C_t + R_t will be the same either way!

- One thing the Lucas model makes clear is that, in general, asset prices will not be random walks. (Contrary to popular belief).

To see this, consider the Euler Eq. →

From Euler,

$$P_t = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}(y_{t+1} + P_{t+1})\right]$$

$$= \beta E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) E_t(y_{t+1} + P_{t+1}) + \beta \, \mathrm{cov}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}, y_{t+1} + P_{t+1}\right]$$

Suppose investors are risk-neutral (u' is constant), then

$$P_t = \beta E_t(y_{t+1} + P_{t+1}) \implies$$ Prices follow a martingale, when adjusted for discounting + dividends

$$\implies P_t = E_t \sum_{j=1}^{\infty} \beta^j y_{t+j}$$ → Price = expected PDV of dividends

Comments

1.) Notice that, in general, asset prices in "efficient markets" are not random walks, due to potential variation in $E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)$ and/or $\mathrm{cov}_t(\cdot)$. This reflects "time-varying risk premia". Prices are martingales only when investors are risk neutral. Hence, predictable movements in asset returns do not (necessarily) reflect market inefficiency!

2.) Technically, a martingale only restricts the (conditional) first moment. Thus, it is less restrictive than a random walk, which restricts the entire distribution. For example, a martingale is consistent with ARCH/GARCH effects in the forecast errors. A random walk is not.

- In the risk-neutral case, we can easily solve for asset prices by evaluating the expected PDV of dividends (given an explicit process for dividends).

- More generally, however, it can be difficult to solve for equilibrium prices, due to potential time-variation in risk premia.

- I will first present 2 special cases where it is possible to obtain closed-form solutions. Then I will present a useful approximation for the general case.

Special Cases

1. Log utility: Imposing the equil. condition $c_t = y_t$ into Euler eq.,

$$P_t u'(y_t) = \beta E_t u'(y_{t+1})(y_{t+1} + P_{t+1})$$
$$= \beta E_t \sum_{j=1}^{\infty} \beta^j u'(y_{t+j}) y_{t+j}$$

Notice that with log util., $u'(y_{t+j}) = \frac{1}{y_{t+j}}$, so that

$$\boxed{P_t = \frac{\beta}{1-\beta} y_t}$$

Question: Why are the dynamics in y_t irrelevant? Why is the P/D ratio constant?
(Hint: Think income + substitution effects).

(2.) <u>Discrete States</u>: Suppose dividends only occupy a finite # of states, $s_1, s_2, \ldots s_n$. Let

$$P_{ij} = \mathrm{Prob}[y_{t+1} = s_j \mid y_t = s_i]$$

We then have the discrete-state Euler eq.

$$p(s_i)u'(s_i) = \beta \sum_{j=1}^{n} p(s_j)u'(s_j)P_{ij} + \beta \sum_{j=1}^{n} s_j u'(s_j) P_{ij}$$

Defining $v_i = p(s_i)u'(s_i)$ and $\alpha_i = \beta \sum_{j=1}^{n} s_j u'(s_j) P_{ij}$,

$$v_i = \alpha_i + \beta \sum_{j=1}^{n} P_{ij} v_j$$

or in vector-matrix notation,

$$\boxed{\begin{aligned} v &= \alpha + \beta P v \\ \Longrightarrow \quad v &= (I - \beta P)^{-1} \alpha \\ \Longrightarrow \quad p_i &= \frac{v_i}{u'(s_i)} \end{aligned}}$$

A Useful Approximation

Write the Euler eq. as follows,

$$1 = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{t+1}\right]$$

where $R_{t+1} = \frac{y_{t+1} + P_{t+1}}{P_t}$ = asset return (not price).

Next, apply "law of iterated expectations" to write in terms of unconditional expectations,

$$1 = E\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{t+1}\right]$$

Specialize to $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ (i.e., CRRA).

$$1 = E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}\right]$$ } a nonlinear function of 2 random variables.

Write this as,

$$1/\beta = E\left[(1+g_{t+1})^{-\gamma}(1+r_{t+1})\right]$$

where g_{t+1} = consumption growth rate

r_{t+1} = asset rate of return

Define the function, $f(g,r) = (1+g)^{-\gamma}(1+r)$ and consider the following 2nd-order Taylor series approx. (around $\bar{g}, \bar{r}$).

$$f(g,r) \approx f(\bar{g},\bar{r}) + \frac{\partial f}{\partial g}(g-\bar{g}) + \frac{\partial f}{\partial r}(r-\bar{r}) + \frac{1}{2}\frac{\partial f^2}{\partial g^2}(g-\bar{g})^2 + \frac{1}{2}\frac{\partial f^2}{\partial r^2}(r-\bar{r})^2 + \frac{\partial f}{\partial g \partial r}(g-\bar{g})(r-\bar{r})$$

Approximating for small growth rates,

$\bar{g} = \bar{r} = 0$

$f(\bar{g},\bar{r}) = 1$

$\frac{\partial f}{\partial g} = -\gamma \qquad \frac{\partial f}{\partial r} = 1$

$\frac{\partial^2 f}{\partial g^2} = \gamma(\gamma+1) \qquad \frac{\partial^2 f}{\partial r^2} = 0 \qquad \frac{\partial^2 f}{\partial g \partial r} = -\gamma$

Sub in,

$$1/\beta \approx E\left[1 + r_{t+1} - \gamma g_{t+1} + \tfrac{1}{2}\gamma(\gamma+1)g_{t+1}^2 - \gamma g_{t+1} r_{t+1}\right]$$

Letting $1/\beta = 1 + \delta$

rate of time preference

$$1+\delta \approx 1 + E(r) - \gamma E(g) + \tfrac{1}{2}\gamma(\gamma+1)\left[\mathrm{var}(g) + \underbrace{(Eg)^2}_{=0}\right] - \gamma\,\mathrm{cov}(g,r) - \underbrace{\gamma E(g)E(g)}_{=0}$$

Eliminating the 2nd-order terms + solving for $E(r)$,

$$E(r) \approx \delta + \gamma E(g) - \tfrac{1}{2}\gamma(\gamma+1)\mathrm{var}(g) + \gamma\,\mathrm{cov}(g,r)$$

Defining

$$\underbrace{r_f \approx \underbrace{\delta}_{\text{impatience}} + \underbrace{\gamma E(g)}_{\text{intertemp. subst.}} - \underbrace{\tfrac{1}{2}\gamma(\gamma+1)\mathrm{var}(g)}_{\text{precautionary saving}}}_{\text{risk-free rate}}$$

We can then write,

$$\boxed{E(r) \approx r_f + \underbrace{\gamma}_{\text{price of risk}}\ \underbrace{\mathrm{cov}(g,r)}_{\text{quantity of risk}}} \quad \text{CCAPM}$$

Letting $\rho = \mathrm{corr}(g,r)$, we can write this as,

$$\boxed{E(r) \approx r_f + \gamma \rho \sigma_g \sigma_r}$$

Everything observable except γ

The Equity Premium Puzzle

- Let's apply this to the case of the U.S. stock market (other countries are similar), so that

$E(r)$: average real return (annual) on U.S. stock mkt.

r_f = average real return on U.S. T-Bills.

$$E(r) - r_f = .07 - .01 = .06$$

$$\sigma_g = .01$$

$$\sigma_r = .16$$

$$\rho = .2$$

$$\Rightarrow \quad \gamma = \frac{.06}{.2(.0016)} = 187\,!$$

(or $\gamma = 37$ if the theoretical value $\rho = 1$ is used)

Equity Premium Puzzle = Very high γ needed to explain observed equity premium

Sources: 1.) Consumption growth smooth relative to stock returns

2.) Consumption growth weakly correlated with stock returns.

Why not high γ?

1.) $\gamma\uparrow$ implies $r_f \uparrow$ (risk-free rate puzzle) unless γ is really big

2.) high γ contradicts observed (and common sense) choices about risk bearing

Calibrating γ

- What share of your wealth are you willing to pay in order to avoid the risk of gaining or losing a share α of it with equal probability? (With CRRA, the answer doesn't depend on level of wealth).

$$\frac{(1-\pi)^{1-\gamma}}{1-\gamma} = \frac{1}{2}\frac{(1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{(1+\alpha)^{1-\gamma}}{1-\gamma}$$

γ	$\alpha = .10$	$\alpha = .3$
$1/2$	0.3%	2.3%
1	0.5%	4.6%
4	2%	16%
10	4.4%	24.4%
40	8.4% !	28.7% !

- It's calculations like these that make many people think that values of γ above 10 are a priori implausible.