Topics for Today

1.) The Equity Premium Puzzle } from last time
- Calibrating Risk Aversion

2.) Hansen - Jagannathan Bounds

3.) Strategies for Resolving the Equity Premium Puzzle
- Data/Measurement Biases
- Exotic Preferences
- Incomplete Markets

Hansen-Jagannathan Bounds

- Whenever an economic model is rejected by the data, an obvious response is to ask: "Maybe I have the wrong utility function".

Usual Strategy

1.) Specify a discount factor model (i.e., utility function)

2.) See if it works

Hansen + Jagannathan's Approach

1.) Derive the properties that any valid discount factor model must satisfy

2.) See if you can devise a theoretically consistent model that meets these requirements.

- That is, H+J pursue a nonparametric, reverse engineering approach.

- Start with, $P(x) = E(mX)$
- With a given utility function [that is date/state separable], we know $m = \beta \frac{u'(c_{t+1})}{u'(c_t)}$
- In what sense does this hold more generally?

1.) Law of One Price $\Longrightarrow$ Pricing functional is linear

mapping from payoff streams to real line

2.) Riesz Rep. Theorem $\Longrightarrow$ Linear functionals in a Hilbert Space have an inner product representation

Let $x \in$ Hilbert Space + $f(x)$ be a linear functional, then we can write $f(x) = y \cdot x$ for some vector y.

3.) Therefore, w.l.o.g, $P(x) = E(mX)$

inner product rep. for pricing functional

Other results,

a.) No arbitrage $\Longrightarrow$ $m > 0$

b.) Complete Mkts. $\Longrightarrow$ m is unique = AD state price density

- Note, m plays 2 important roles

$$\boxed{P(x) = E(mx) = \underbrace{E(m)}_{\text{time discount}}E(x) + \underbrace{cov(m,x)}_{\text{risk}}}$$

- Also note,

$P(1) = E(m)$ = Price of sure claim
= reciprocal of (gross) riskless interest rate

Simple Example - Excess Returns

$$1 = E(mR)$$
$$1 = E(m\underbrace{R_f}_{\text{risk free rate}})$$
$$\Longrightarrow \quad 0 = E(mR^e)$$

where $R^e = R - R_f$: "excess return"

Therefore,

$$E(m)E(R^e) + cov(m, R^e) = E(m)E(R^e) + \rho\,\sigma_m\,\sigma_{R^e}$$
$$= 0$$

Re-arranging,

$$\frac{E(R^e)}{\sigma(R^e)} = -\rho \frac{\sigma_m}{E(m)}$$

Since $|\rho| \leq 1$, we have

$$\boxed{\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma_m}{E(m)}} \Bigg\} \text{ lower bound on discount factor volatility}$$

Sharpe Ratio

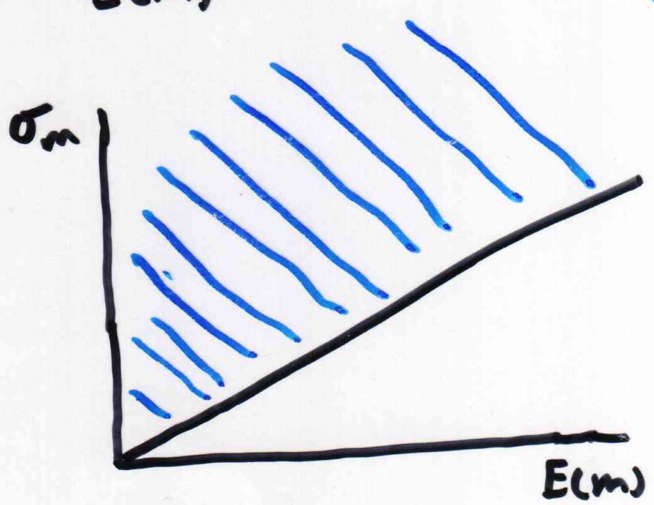
In U.S. Data, $E(R^e) \approx .06$, $\sigma(R^e) \approx .16$, thus

$$\frac{.06}{.16} = .375 < \frac{\sigma_m}{E(m)}$$

<u>Multiple Assets</u>: $\frac{\sigma_m}{E(m)} \geq \left[E(R^e)' \Sigma^{-1} E(R^e)\right]^{1/2}$

Σ = var-cov matrix of R^e

<u>Graphically</u>:



- Now extend to returns (rather than excess returns)

$$\underset{n\times 1}{1} = E(\underset{n\times 1}{R} \cdot \underset{1\times 1}{m})$$

- Note that $m^* = R'[E(RR')]^{-1} \cdot 1$ provides a valid discount factor for these given assets. Also note that $m = m^* + \varepsilon$, where ε is any random variable uncorrelated with R, also provides a valid discount factor.

Question: Which linear combo. of R delivers the minimum variance discount factor?

Define: $\mu_m = E(m)$ $\quad \mu_R = E(R)$ $\quad \Sigma = E(R-\mu_R)(R-\mu_R)'$

and project $m - \mu_m$ onto $R - \mu_R$

$$m - \mu_m = (R - \mu_R)'\beta + u$$

where $\hat{\beta} = \Sigma^{-1} E(R-\mu_R)(m-\mu_m)$ > Least squares Formula

From the pricing equation,

$$E(R-\mu_R)(m-\mu_m) = E(R\cdot m) - \mu_R \mu_m$$
$$= 1 - \mu_R \mu_m$$

$$\Rightarrow \hat{\beta} = \Sigma^{-1}(1 - \mu_R \mu_m)$$

By construction, u is orthogonal to regressors. Therefore,

$$\sigma_m^2 = E[\hat{\beta}'(R-\mu_R)(R-\mu_R)'\hat{\beta}] + \sigma_u^2$$
$$= (1-\mu_R\mu_m)'\Sigma^{-1}\Sigma\Sigma^{-1}(1-\mu_R\mu_m) + \sigma_u^2$$

Therefore,

$$\boxed{\sigma_m^2 \geq \sqrt{(1-\mu_R\mu_m)'\Sigma^{-1}(1-\mu_R\mu_m)}} \Big\} \text{Hansen Jagannathan Bound}$$

- This defines a parabolic region in $(E(m), \sigma_m)$ space, which depends on the means and var-cov matrix of observed returns.
- Let's look at a couple examples

Hansen-Jagannathan Bound

U.S. Stock Mkt. + Treasury Bills

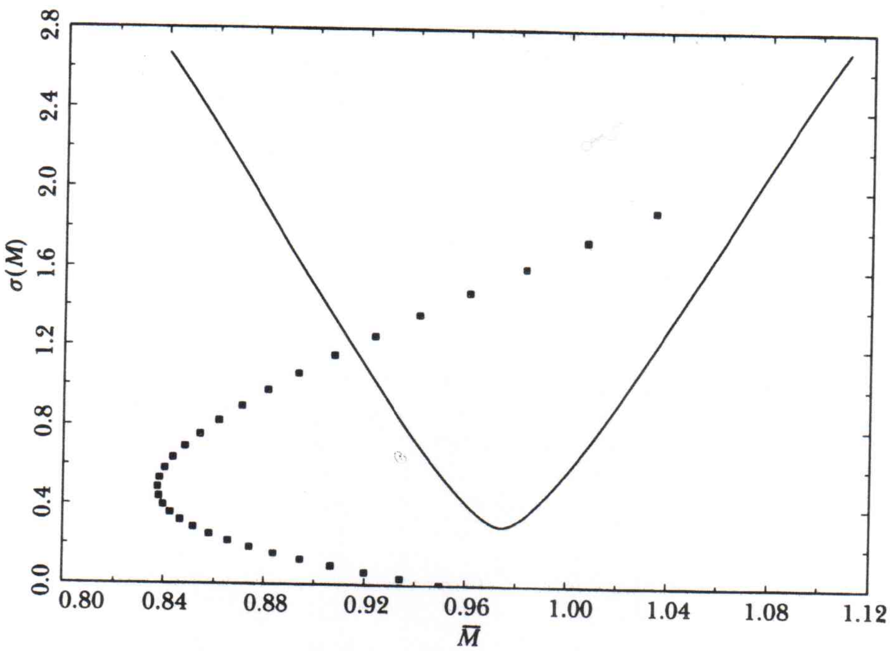


Figure 8.3. Feasible Region for Stochastic Discount Factors Implied by Annual US Data, 1891 to 1994

- Solid squares depict implied σ_m, $E(m)$ for traditional time + state separable CRRA preferences,

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

for $\beta = .98$ and $\gamma = 1, 2, 3, \ldots\ldots 31$

Foreign Exchange Market

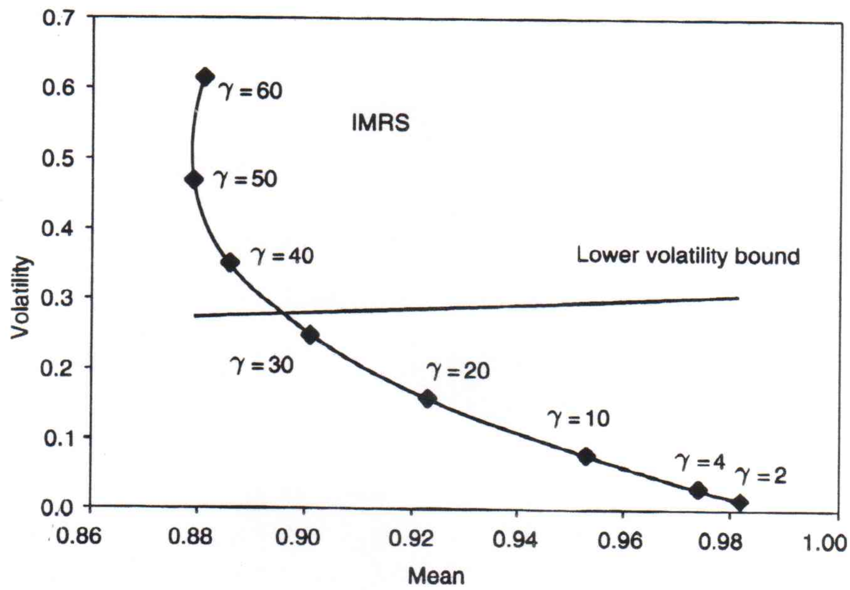


Figure 6.2 Mean and volatility estimates of the intertemporal marginal rate of substitution (IMRS) with $\beta = 0.99$ and alternative values of γ under constant relative risk-aversion utility and a lower bound implied by forward exchange payoffs of the pound, deutschemark, and yen, 1973.1 to 1997.1.

$$0 = E\left[\left(\frac{F_t - S_{t+1}}{S_t}\right) \cdot m\right]$$

F_t = forward rate

S_t = spot ex. rate

$F_t - S_{t+1}$ = profit from buying forward contract

Strategies for Resolving the Equity Premium Puzzle

1.) Data / Measurement Biases

- Taxes (McGrattan + Prescott (AER, 2003))
- Survivorship Bias (Brown, Goetzmann + Ross (JF 1995))
- Rare Disasters / Peso Problems (Barro (QJE (2006))

2.) Exotic Preferences

- Date + State Nonseparabilities
  (Habit Persistence + Epstein-Zin)
- Ambiguity / Model Uncertainty

3.) Incomplete Markets

- Persistent Idiosyncratic Labor Income Risk
  (Mankiw (1986), Constantinides + Duffie (1996))

Exotic Preferences

- The traditional CRRA specification, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, constrains relative risk aversion to be the reciprocal of the intertemporal elasticity of substitution. Why should substitution across states (risk aversion) have anything per se to do with substitution across dates (intertemp. subst.).

- A general way to express dynamic utility function is in the following recursive form:

$$\boxed{U_t = V\left(u(c_t), \mu_t(U_{t+1})\right)}$$

$V(\cdot)$: Time Aggregator

$\mu(\cdot)$: Certainty Equivalent Function

- Traditional Specification is the special case,

$$V = u(c_t) + \beta \mu_t$$

$$\mu_t = E_t U_{t+1}$$

- Epstein + Zin (1989) proposed

$$V = \left[(1-\beta)u(c)^{\rho} + \beta \mu(U)^{\rho}\right]^{1/\rho}$$

$$\mu(U) = \left[E(U^{\alpha})\right]^{1/\alpha}$$

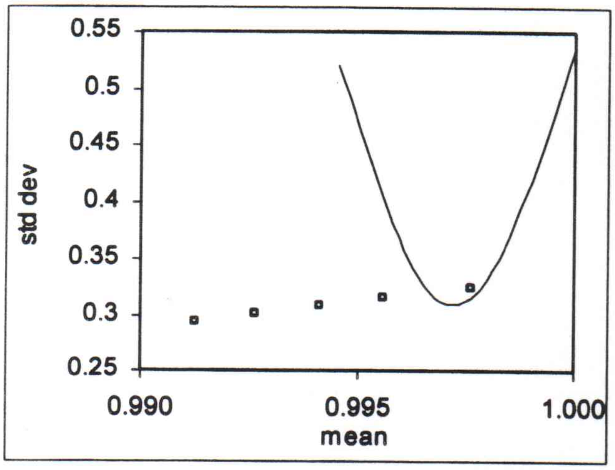
$1-\alpha$: risk aversion

$\frac{1}{1-\rho}$: intertemp. elast. of subst.

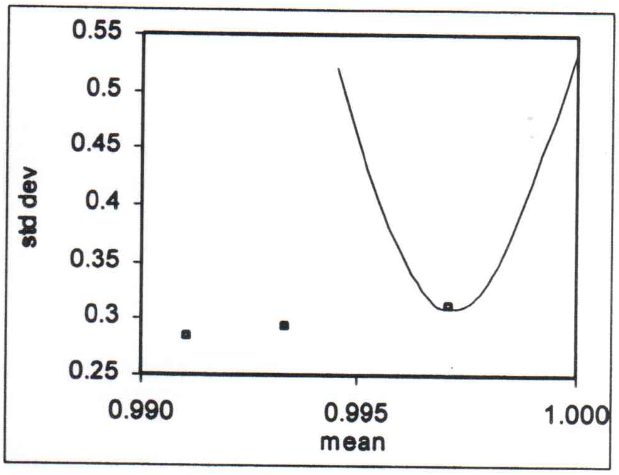
$$\boxed{\alpha = \rho \Rightarrow \text{Traditional CRRA}}$$

Resolutions of the Asset Pricing Puzzle, 1947-1997

a: Epstein-Zin



b: Habit Formation



$$m_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\right]^{\frac{1-\gamma}{1-\sigma}} \left(R_{t+1}\right)^{\frac{\sigma-\gamma}{1-\sigma}}$$

$$\cancel{m_{t+1} = \beta \, (C_{t+1} - \delta C_t)^{-\gamma} + \beta\delta E_{t+1} (C_{t+2}}$$

IES = $1/\sigma$

RRA = γ

R_{t+1} = mkt. return

$\beta = .99$

$\sigma = 3.15$

$\gamma \in [15, 16.4]$

$\beta = .96$

$\delta = .72$

$\gamma \in [3.0, 3.3]$

$$m_{t+1} = \beta \, \frac{(C_{t+1} - \delta C_t)^{-\gamma} + \beta\delta E_{t+1} (C_{t+2} - \delta C_{t+1})^{-\gamma}}{(C_t - \delta C_{t-1})^{-\gamma} - \beta\delta E_t (C_{t+1} - \delta C_t)^{-\gamma}}$$

Incomplete Markets

- Researchers long suspected that incomplete mkts. were responsible for the EPP (Mehra & Prescott (1985) conjectured this in the Conclusion to their paper).

- However, early attempts to explain the EPP with incomplete mkts. seemed trapped by 2 facts:

1.) Pure idiosyncratic risk doesn't affect asset prices:

$$\frac{E(r) - r_f}{\sigma_r} = \gamma \cdot \rho_{\Delta c, r} \cdot \sigma_{\Delta c}$$

$$\sigma_{\Delta c} \uparrow \implies \rho \downarrow$$

Thus, it seems "idiosyncratic" consumption risk must be correlated with returns.

2.) Unless idiosyncratic income shocks are permanent, individuals can use assets mkts. to smooth their effects on consumption.

- Constantinides & Duffie (JPE, 1996) showed how persistent idiosyncratic labor income risk could explain the EPP.

Constantinides & Duffie

- I will first give a special case that illustrates the basic idea. Then I give the general case.

Example

$$U = E \sum_t e^{\delta t} C_{it}^{1-\gamma}$$

$$\ln\left(\frac{C_{it+1}}{C_{it}}\right) = \eta_{it+1} y_{t+1} - \frac{1}{2} y_{t+1}^2$$

$$\eta_{it} \sim N(0,1)$$ } idiosyncratic shock to consumption (and income)

y_t = st. dev. of cross-sectional dist. of consumption growth

Therefore, conditional on y_{t+1},

$$\ln\left(\frac{C_{it+1}}{C_{it}}\right) \sim N\left(-\frac{1}{2} y_{t+1}^2, y_{t+1}^2\right)$$

$$\Rightarrow E\left(\frac{C_{it+1}}{C_{it}}\right) = 1$$

Assume, $\boxed{y_{t+1} = \sqrt{\frac{2}{\gamma(\gamma+1)}} \cdot \sqrt{\delta - \ln R_{t+1}}}$

Thus, $\boxed{R_{t+1} \downarrow \Rightarrow \text{cross-sectional variance} \uparrow}$

- From individual Euler eq.,

$$1 = E_t\left[e^{-\delta}\left(\frac{c_{it+1}}{c_{it}}\right)^{-\gamma} R_{t+1}\right]$$

$$= E_t\left[e^{-\delta}\left(e^{\eta_{it+1} y_{t+1} - \frac{1}{2} y_{t+1}^2}\right)^{-\gamma} R_{t+1}\right]$$

$$= E_t\left[e^{-\delta}\, e^{-\gamma \eta_{it+1} y_{t+1} + \frac{1}{2}\gamma y_{t+1}^2}\, e^{\ln R_{t+1}}\right]$$

$$= E_t\left[e^{-\delta - \gamma \eta_{it+1} y_{t+1} + \frac{1}{2}\gamma y_{t+1}^2 + \ln R_{t+1}}\right]$$

Note: $E[f(\eta,y)] = E[E(f(\eta,y)|y)]$

$$\Rightarrow E\left[e^{-\gamma \eta_{it+1} y_{t+1}} \,\middle|\, y_{t+1}\right] = e^{\frac{1}{2}\gamma^2 y_{t+1}}$$

Therefore,

$$1 = E_t\left[e^{-\delta + \frac{1}{2}\gamma^2 y_{t+1} + \frac{1}{2}\gamma y_{t+1}^2 + \ln R_{t+1}}\right]$$

$$= E_t\left[e^{-\delta + \frac{1}{2}\gamma(\gamma+1)\frac{2}{\gamma(\gamma+1)}(\delta - \ln R_{t+1}) + \ln R_{t+1}}\right]$$

$$= E_t\left[e^0\right] = 1 \quad !$$

- Thus, CD reverse engineer a discount factor process, based on idiosyncratic risk, which by construction solves the EPP.

General Case

g_{it} = gross rate of consumption growth for agent i
$\left(= \frac{c_{it}}{c_{it-1}}\right)$

$M(g_{it})$ = agent i's stochastic discount factor
$(= \beta g_{it}^{-\gamma}$ for CRRA).

Let F_t = time-t cross-sectional dist. of g_{it}
(i.e., $F_t(G) = \text{prob}(g_{it} \leq G)$)

Define the following moments,

$$\mu_{1t} = \int g \, F_t(dg)$$ > time-t cross-sectional mean

$$\mu_{2t} = \int (g - \mu_{1t})^2 F_t(dg)$$ > time-t cross-sectional variance

Expand $M(g_{it})$ around μ_{1t}

$$\boxed{M(g_{it}) = M(\mu_{1t}) + M'(\mu_{1t})(g_{it} - \mu_{1t}) + \frac{1}{2} M''(\mu_{1t})(g_{it} - \mu_{1t})^2}$$

Individual Euler Eqs.

$$E[R_t M(g_{it})] = 1$$

$$E[R^e_t M(g_{it})] = 0$$

Sub in Taylor series approx.,

$$E\left\{R_t\left(M(\mu_{1t}) + M'(\mu_{1t})(g_{it}-\mu_{1t}) + \tfrac{1}{2}M''(\mu_{1t})(g_{it}-\mu_{1t})^2\right)\right\} = 1$$

$$\Rightarrow E\left\{R_t\left(M(\mu_{1t}) + \tfrac{1}{2}M''(\cdot)\mu_{2t}\right)\right\} = 1$$

Remember, riskless rate = reciprocal of mean disc. rate

$$E R_{ft} = \frac{1}{E[M + \frac{1}{2}M'' \cdot \mu_{2t}]}$$

Therefore,

$$\boxed{E R^e_t = \frac{-\operatorname{cov}\left(R^e_t, M + \frac{1}{2}M''\mu_{2t}\right)}{E\left(M + \frac{1}{2}M'' \cdot \mu_{2t}\right)}}$$

Note if $M'' > 0$ and $\mu_{2t}\uparrow$ when $R^e_t \downarrow$,

then $E R^e \uparrow$