Topics for Today

1.) Fourier Transforms + Spectral Densities (from last time).

2.) A little Functional Analysis
- a hierarchy of spaces
- convergence + completeness

3.) Contraction Mappings + Fixed Points
- Theorem of the Maximum
- Blackwell's Sufficient Conditions

A Little Functional Analysis

- We want to solve

$$\max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \quad \text{s.t. } x_{t+1} = g(x_t, u_t)$$
$$x_0 \text{ given}$$

where x_t is the state. To do this, we define the value function,

$$V(x_0) = \max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \quad \text{s.t. } x' = g(x, u)$$

- Exploiting the recursive structure of this problem,

$$\boxed{V(x) = \max_u \{ r(x,u) + \beta V(x') \} = \max_u \{ r(x,u) + \beta V(g(x,u)) \}}$$

- Note, if we know V, we can solve for the optimal policy as,

$$u = h(x) = \operatorname*{argmax}_u \{ r(x,u) + \beta V[g(x,u)] \}$$

- We then have the identity characterizing the value + policy functions

$$\boxed{V(x) = r(x, h(x)) + \beta V[g(x, h(x))]}$$

- These are <u>functional equations</u>. ["Functions of functions."] We will analyze them using the tools of "functional analysis"

- As a first step, define the r.h.s. of the Bellman eq. as follows:

$$T(V) \equiv \max_u \{ r(x,u) + \beta V[g(x,u)] \}$$

- Note, T is an <u>operator</u> (i.e., it maps elements from one vector space to another).

- We can now write the Bellman Eq. as,

$$\boxed{V = T(V)}$$

- Solving the Bellman Eq. is equivalent to finding a <u>fixed point</u> of T.

- Recasting a problem as a Fixed Point problem is a common strategy in applied mathematics.

Examples

1.) Algebraic Eqs.

Want to find the roots of $f(x)=0$

Define $T(x) = x + f(x)$.

Then solving $f(x)=0$ is the same as finding the fixed pts. of $x = T(x)$.

2.) Differential Eqs.

Want to solve $\dot{x} = f(x)$

Integrate both sides, $x(t) = x(0) + \int_0^t f(x(s))ds$

$$= T(x)$$

3.) Competitive Equilibrium

Arrow & Debreu (1954) provided the first rigorous proof of the existence of a competitive equil. by posing the problem as a FP problem, and then using Kakutani's FP theorem.

(Earlier attempts had just counted eqs. & unknowns!)

A Hierarchy of Spaces

(1.) **Topological Space**: A set endowed with a "topology", which is a family of subsets called "open sets" which is closed under finite intersections and countable unions, and contains a null element (empty set) and the whole space.

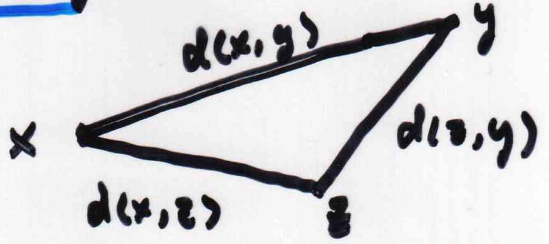
(2.) **Metric Space**: A topological space where the topology is induced by a distance function. A distance function is a set X and a real-valued function $d: X \times X \to R$, called a metric, which satisfies 4 properties

a.) Positivity: $d(x,y) \geq 0 \quad \forall\, x,y \in X$

b.) Strict Positivity: $d(x,y) = 0$ iff $x = y$

c.) Symmetry: $d(x,y) = d(y,x)$

d.) Triangle Inequality: $d(x,y) \leq d(x,z) + d(z,y)$



Examples

1.) Let $X = \ell_p[0,\infty)$ [set of $\{x_t\}$ s.t. $\sum_{t=0}^{\infty} |x_t|^p < \infty$]

Then $d_p(x,y) = \left(\sum_{t=0}^{\infty} |x_t - y_t|^p \right)^{1/p}$

2.) Let $X = C[0,T]$ (set of continuous real-valued functions on $[0,T]$)

Then $d_p(x,y) = \left[\int_0^T |x(t) - y(t)|^p \, dt \right]^{1/p}$

or $d_\infty(x,y) = \sup_{0 \le t \le T} |x(t) - y(t)|$

- We now want to add some algebraic structure,

3.) **Linear (vector) Space**: A set of elements closed under addition and scalar multiplication, which satisfies the usual algebraic axioms (commutative, associative, distributive, etc).

Consider $C[0,T]$ + note

$$(x+y)(t) = x(t) + y(t)$$
$$(\lambda x)(t) = \lambda x(t)$$

④ **Normed Linear Space** : A linear space, X, along with a real-valued function that maps $x \in X$ into a real-number, $\|x\|$, which satisfies the properties :

a.) $\|x\| \geq 0 \quad \forall x \in X \qquad \|x\| = 0$ iff $x = 0$

b.) $\|\lambda x\| = |\lambda| \cdot \|x\| \qquad \forall$ scalars λ

c.) $\|x+y\| \leq \|x\| + \|y\|$

Note : A norm can be interpreted as a function space analog of the notion of the "size" or "modulus" of a vector.

Comments

1.) A norm defines a metric, $d(x,y) = \|x-y\|$

2.) A given space, X, can have many different norms. Technically, these define different normed vector spaces. A normed space consists of both a set of elements and the norm.

3.) In finite dimensional spaces, all norms are topologically equivalent ($\|\cdot\|_1$ is top. equiv. to $\|\cdot\|_2$ if $\exists$ positive numbers a and b s.t. $a\|x\|_1 < \|x\|_2 < b\|x\|_1 \quad \forall x \in X$).

Convergence + Completeness

- Our solution algorithms will be iterative. They will produce a sequence of successively better approximations. We would like to know if the sequence converges, and if so, what sort of object it converges to.

Convergence

A sequence $\{x_n\}$ in a metric space (X, d) converges to a limit $x_0 \in X$ if for every $\varepsilon > 0$ $\exists\, N(\varepsilon)$ s.t. $d(x_n, x_0) < \varepsilon \quad \forall\, n > N(\varepsilon)$.

- Note, to establish convergence here we must already know the limit pt. Not of much use to us! A more useful concept is the following:

Cauchy Sequence

A sequence $\{x_n\}$ in a metric space (X, d) is a Cauchy sequence if for each $\varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t. $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N(\varepsilon)$ (i.e., $\lim\limits_{n,m \to \infty} d(x_n, x_m) = 0$).

Fact 1 : Convergent sequences are Cauchy Sequences

(Proof : Apply $\Delta \neq$, $d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m)$).

Fact 2 : Cauchy sequences may not converge (i.e., may not have a limit in X).

Famous Example : X: Rational Numbers, $d(x,y) = |x-y|$

More Relevant Example : $X = C[0,1]$ with $d_2(x,y)$

consider $x_n(t) = t^n$. Then

$$d_2(t^m, t^n)^2 = \int_0^1 (t^m - t^n)^2 dt$$

$$= \frac{1}{2m+1} + \frac{1}{2n+1} - \frac{2}{m+n+1}$$

Note: x_n is Cauchy, but its limit is not in $C[0,1]$.

Completeness

A metric space is complete if each Cauchy sequence in (X, d) converges to an element of X.

- Completeness is important because when solving functional eqs. numerically, we do so by generating a sequence of approximations. We want to know that the limit is in the relevant set.
- Completeness + Cauchy allows you to prove convergence without knowing the limit!
- There are 2 strategies for getting completeness:

1.) Change the space (i.e., include limits of Cauchy seqs.)

Famous Examples : a.) Real vs. Rational numbers

b.) $L_2[0,T]$ vs. $C_2[0,T]$.

2.) Change the norm

Example: $C[0,T], d_2$ is not complete

but $C[0,T], d_\infty$ is complete

uniform convergence preserves continuity

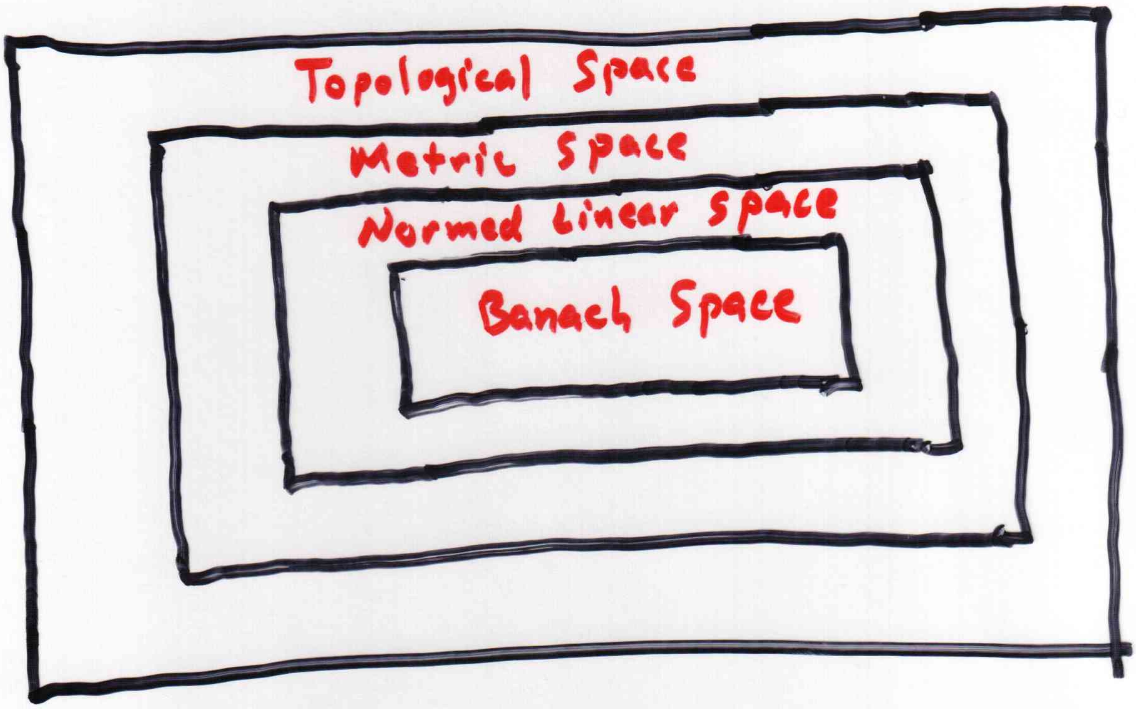
(Note: $t^n = x_n(t)$ is not Cauchy under d_∞).

- We have now come to the end of the line. We have,

Definition

Banach Space : A complete normed linear space

- We can visualize what we've accomplished as follows:



- Actually, this isn't quite the end of the story. For some problems it is useful to impose some geometric structure that allows us to define things like orthogonal projections. This leads to the concept of a Hilbert Space. A Hilbert space is a Banach space endowed with an inner product.

The Contraction Mapping Theorem

- The Contraction Mapping Theorem is sometimes referred to as the "Banach Fixed Pt. Theorem."

- First some more jargon,

Operator : A mapping from one vector space to another

Linear Operator : $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$

Continuous Operator : $d(T(x),T(x_0))<\varepsilon$ whenever $d(x,x_0)<\delta$
(or $\lim x_n \to x_0 \Longrightarrow \lim T(x_n) \to T(x_0)$).

Definition

Let (X,d) be a metric space and let $T: X \to X$. We say T is a contraction mapping if $\exists\ 0 \leq k < 1$ s.t. $d[T(x),T(y)] \leq k\,d(x,y)$ $\forall\, x,y \in X$.

Note: Contraction Mappings are continuous (let $\delta = \varepsilon/k$).

Contraction Mapping Theorem

Let (X, d) be a complete metric space, and let $T: X \to X$ be a contraction. Then T has a unique fixed point, $T(x^*) = x^*$, which is the limit of the recursion, $x_{n+1} = T(x_n) = T^n(x_0)$.

- Note, the Contraction Mapping Th. is an unusual FP theorem, for two reasons:
  a.) It delivers uniqueness
  b.) It provides an algorithm for finding the fixed pt.

Proof: The proof consists of 3 steps:

1.) Show $x_n = T^n(x_0)$ is Cauchy

2.) Prove that the limit, x_∞, is a FP of T.

3.) Establish uniqueness

Step 1

$$d(x_{m+1}, x_m) = d(T(x_m), T(x_{m-1})) \quad \text{> defn. of } T$$
$$\leq K\, d(x_m, x_{m-1}) \quad \text{> contraction property}$$
$$= K\, d(T(x_{m-1}), T(x_{m-2})) \quad \text{> defn. of } T$$
$$\leq K^2 d(x_{m-1}, x_{m-2}) \quad \text{> contraction property}$$
$$\vdots$$
$$\leq k^m d(x_1, x_0)$$

Assume $n > m$. Then $\Delta \neq$ implies,

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots d(x_{m+1}, x_m)$$
$$\leq (K^{n-1} + K^{n-2} + \cdots + K^{m+1} + K^m)\, d(x_1, x_0)$$
$$= k^m \frac{1-k^{n-m}}{1-k} d(x_1, x_0) \leq \frac{K^m}{1-K} d(x_1, x_0)$$

Thus, we can make $d(x_n, x_m)$ as small as we want by $m \to \infty$. Because X is complete x_n converges to an element of X.

Step 2

Let $x_\infty = \lim_{n \to \infty} x_n = T^n(x_0)$

Want to show $x_\infty = T(x_\infty)$

$$d(x_\infty, T(x_\infty)) \leq d(x_\infty, x_m) + d(x_m, T(x_\infty))$$ > $\Delta \neq$

$$\leq \underbrace{d(x_\infty, x_m)}_{=0} + K \underbrace{d(x_{m-1}, x_\infty)}_{=0}$$ > contraction property

> convergence

Step 3

Prove uniqueness by contradiction.

Suppose x_∞ and y_∞ are ~~a~~ each fixed pts.

$$d(x_\infty, y_\infty) \underset{\text{defn. of FP}}{=} d(T(x_\infty), T(y_\infty)) \leq K d(x_\infty, y_\infty)$$

$$< d(x_\infty, y_\infty)$$

$\Longrightarrow$ contradiction

Blackwell's Sufficient Conditions

- To apply the Contraction Mapping Theorem, we need an easy way to identify a contraction. This is provided by Blackwell's Theorem:

Blackwell's Sufficient Conditions

Let T be an operator on a metric space (X, d_∞). Assume T satisfies:

Note use of sup norm

1.) Monotonicity: $x \geq y \Longrightarrow T(x) \geq T(y)$
$[x \geq y$ iff $x(t) \geq y(t) \quad \forall t]$.

2.) Discounting: Let c be a constant function. For any positive real c and every $x \in X$, $T(x+c) \leq T(x) + \beta c$ for some $0 \leq \beta < 1$

Then T is a contraction with modulus β.

Proof of Blackwell

Let $x = y + x - y$

Note, $x - y \leq d_\infty(x, y)$ > By defn. of sup norm

$\Longrightarrow x \leq y + d_\infty(x, y)$

By monotonicity of T,

$$T(x) \leq T[y + d_\infty(x, y)]$$

By discounting property,

$$T(x) \leq T(y) + \beta d_\infty(x, y)$$

$$\Longrightarrow T(x) - T(y) \leq \beta d_\infty(x, y)$$

By reversing roles of x + y,

$$T(y) - T(x) \leq \beta d_\infty(x, y)$$

$$\Longrightarrow d_\infty(T(x), T(y)) \leq \beta d_\infty(x, y)$$

Application of Blackwell to Bellman Equation

Suppose $V \geq W$

$$T(V) = \max_u \{ r(x,u) + \beta V(x') \}$$

$$\geq \max_u \{ r(x,u) + \beta W(x') \}$$

$$= T(W)$$

$\Longrightarrow$ T is monotone

$$T(V+c) = \max_u \{ r(x,u) + \beta [V(x') + c] \}$$

$$= \max_u \{ r(x,u) + \beta V(x') \} + \beta c$$

$$= T(V) + \beta c$$

$\Longrightarrow$ T has discount property