Topics for Today

1.) Solving Expectational Difference Equations
   - Bubbles + Sunspots
   - The Blanchard-Kahn Method

2.) McCall's Job Search Model

Solving Expectational Difference Equations

- The equilibrium conditions of many economic models take the form of "expectational difference equations". Examples include Linear Quadratic models of consumption + asset pricing:

$$\boxed{x_t = a E_t x_{t+1} + b \underbrace{z_t}_{\text{exogenous}}}$$

- How do we solve this? As always,

$$\boxed{\text{General Sol.} = \text{Particular Sol.} + \text{Homogeneous Sol.}}$$

- We can obtain the particular solution by a "guess + verify" strategy (i.e., "method of undetermined coefficients").

- Suppose $z_t = \rho z_{t-1} + \varepsilon_t$. Given this, let's guess

$$x_t = \beta z_t$$

where β is to be determined.

- With this guess,

$$E_t x_{t+1} = E_t \beta z_{t+1} = \beta \rho z_t$$

Substituting this into the equation, we get the identity:

$$\beta z_t \equiv a \beta \rho z_t + b z_t$$

$$\Rightarrow \boxed{\beta = \frac{b}{1 - a\rho}}$$

- In economics, the particular soln. $x_t = \frac{b}{1-a\rho} z_t$ is often called the "fundamental soln", to remind you & emphasize that it depends on economic "fundamentals".

- Let's write the general soln. as,

$$x_t = f_t + B_t$$

where f_t is the fundamental soln., and B_t is a solution of the homogeneous eq.

- It turns out that the nature of the solutions to homogeneous eq. depend crucially on the value of a.

- The homogeneous eq. is,

$$x_t = a E_t x_{t+1}$$

There are 2 cases to consider,

Case 1 : $|a| < 1$

Consider the process

$$B_t = \frac{1}{a} B_{t-1} + V_t$$

where V_t is any "martingale difference" sequence (i.e., $E_t V_{t+1} = 0$). Note that if we sub this into the homog. eq. we get,

$$B_t = a E_t B_{t+1} = a(1/a) B_t = B_t$$

Hence, $B_t = \frac{1}{a} B_{t-1} + V_t$ solves the homog. eq. However, note also that since $|a|<1$ this solution is explosive. For this reason, it is often called a bubble.

- Sometimes, when the expectational diff. eq. arises from an optimization problem, a TVC rules out this kind of bubble. However, in other cases (e.g., models of fiat money), bubbles cannot be ruled out.

- Note, even when $|a|<1$, B_t might not be explosive if it can occasionally "collapse". Consider the process [Blanchard (1979)]:

$$B_{t+1} = \frac{1}{a\pi} B_t + V_{t+1} \quad \text{w.p. } \pi$$

$$= V_{t+1} \quad \text{w.p. } (1-\pi)$$

Note that $E_t B_{t+1} = \frac{1}{a} B_t$, as before.

Case 2 : $|a| > 1$

We can alway write the decomposition,

$$B_{t+1} = E_t B_{t+1} + V_{t+1}$$

where again, V_{t+1} is a martingale difference. Sub this into the homog. eq.,

$$B_t = a\left[B_{t+1} - V_{t+1}\right]$$

$$\Longrightarrow \quad B_{t+1} = \tfrac{1}{a} B_t + V_{t+1}$$

as before. Now, however, since $|a|>1$, this process is stationary, not explosive. In cases like this, B_t is called a sunspot process.

- Therefore, when $|a|>1$, the solution is indeterminate. There are multiple stationary equilibria.

- To summarize - for the eq. $X_t = a E_t X_{t+1} + b Z_t$,

  $|a|<1 \Rightarrow$ unique stationary solution

  $|a|>1 \Rightarrow$ multiple stationary solutions

- The method of Blanchard-Kahn generalizes this result to the vector case. Now a becomes an nxn matrix, A, and b becomes a nxk matrix, B. Now questions of existence as well as uniqueness arise. The key condition depends on $\hat{A}$ having the "right" number of stable + unstable eigenvalues.

McCall's Model

- McCall's model is one of the workhorse models of unemployment. As an example of DP, it is very simple, as there are only 2 discrete actions each period.

Assumptions

1.) Each period a worker draws an offer, w, from the same wage distribution $F(W) = \Pr[w \le W]$. $[F(0)=0 \quad F(B)=1]$

2.) If the worker rejects, he gets c this period in unemployment compensation, and then draws a new wage next period

3.) If the worker accepts the current wage, w, he receives w each period, forever.

4.) No quitting or firing

5.) No recall of past offers

6.) Worker maximizes $E \sum_{t=0}^{\infty} \beta^t y_t$, where

$$y_t = c \text{ if unemployed}$$
$$= w \text{ if employed}$$

risk neutrality

- Let $V(w) = E\sum \beta^t y_t$ for a worker who has w in hand, and behaves optimally from now on,

Bellman Equation

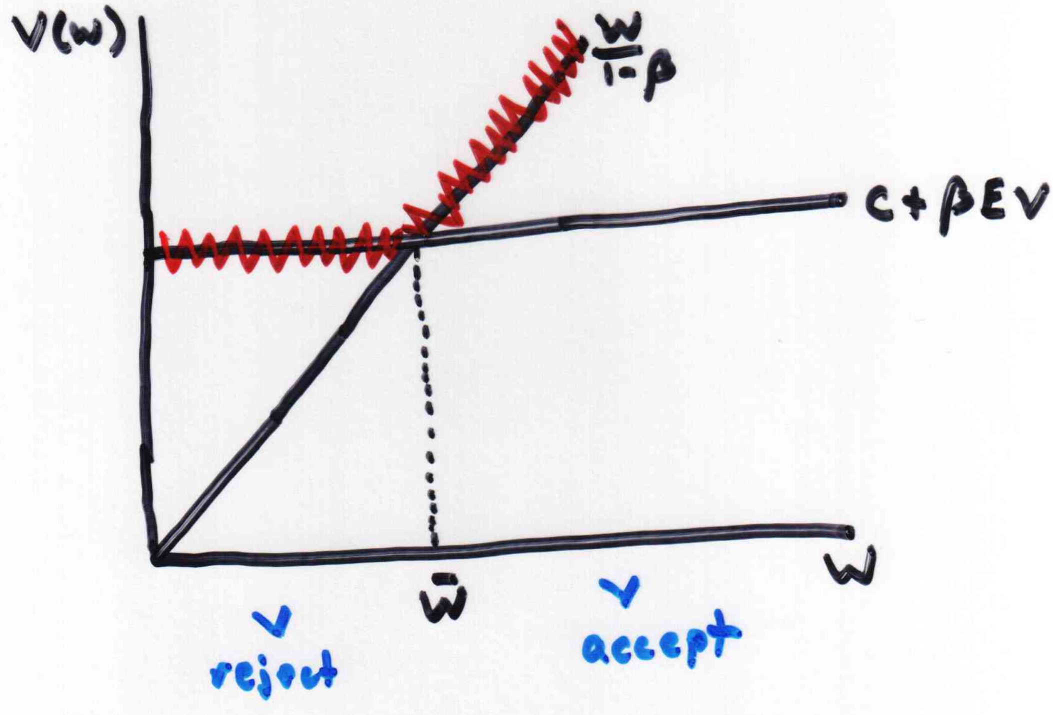
$$\boxed{V(w) = \max\left\{\underbrace{\frac{w}{1-\beta}}_{\text{accept}},\ \underbrace{c + \beta\int_0^B V(w')\,dF(w')}_{\text{reject}}\right\}}$$

2 key observations

1.) First part is linearly increasing in w

2.) Second part is independent of w

Hence we can depict the Bellman Eq. as follows:



- From inspection,

$$V(w) = \begin{cases} w/1-\beta & \text{if } w \geq \bar{w} \\ c+\beta EV & \text{if } w \leq \bar{w} \end{cases}$$

$$c+\beta EV = \frac{\bar{w}}{1-\beta}$$

- We can use this to convert the functional eq. in V to an algebraic eq. in $\bar{w}$.

First, by definition

$$\frac{\bar{w}}{1-\beta} = c + \beta\int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \beta\int_{\bar{w}}^{B} \frac{w'}{1-\beta} dF(w')$$

$$\frac{\bar{w}}{1-\beta}\left[\int_0^{\bar{w}} dF(w') + \int_{\bar{w}}^{B} dF(w')\right] = c + \beta\int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \beta\int_{\bar{w}}^{B} \frac{w'}{1-\beta} dF(w')$$

Collect common integrals,

$$\bar{w}\int_0^{\bar{w}} dF(w') - c = \frac{1}{1-\beta}\int_{\bar{w}}^{B}(\beta w' - \bar{w})\, dF(w')$$

Finally, add $\bar{w}\int_{\bar{w}}^{B} dF(w')$ to both sides

$$\boxed{\underbrace{\bar{w} - c}_{\text{cost of search given } \bar{w}} = \underbrace{\frac{\beta}{1-\beta}\int_{\bar{w}}^{B}(w'-\bar{w})\, dF(w')}_{\text{expected benefit of search given } \bar{w}}}$$

Comparative Statics

- Let's consider two comp. static experiments:

1.) An increase in c

2.) A mean-preserving spread of the wage distribution

Define,

$$h(w) = \frac{\beta}{1-\beta}\int_{w}^{B}(w'-w)\,dF(w')$$

Note, $h(0) = \frac{\beta}{1-\beta}E(w) \quad h(B) = 0$

Also note,

$$h'(w) = \frac{-\beta}{1-\beta}\left(1-F(w)\right) < 0$$

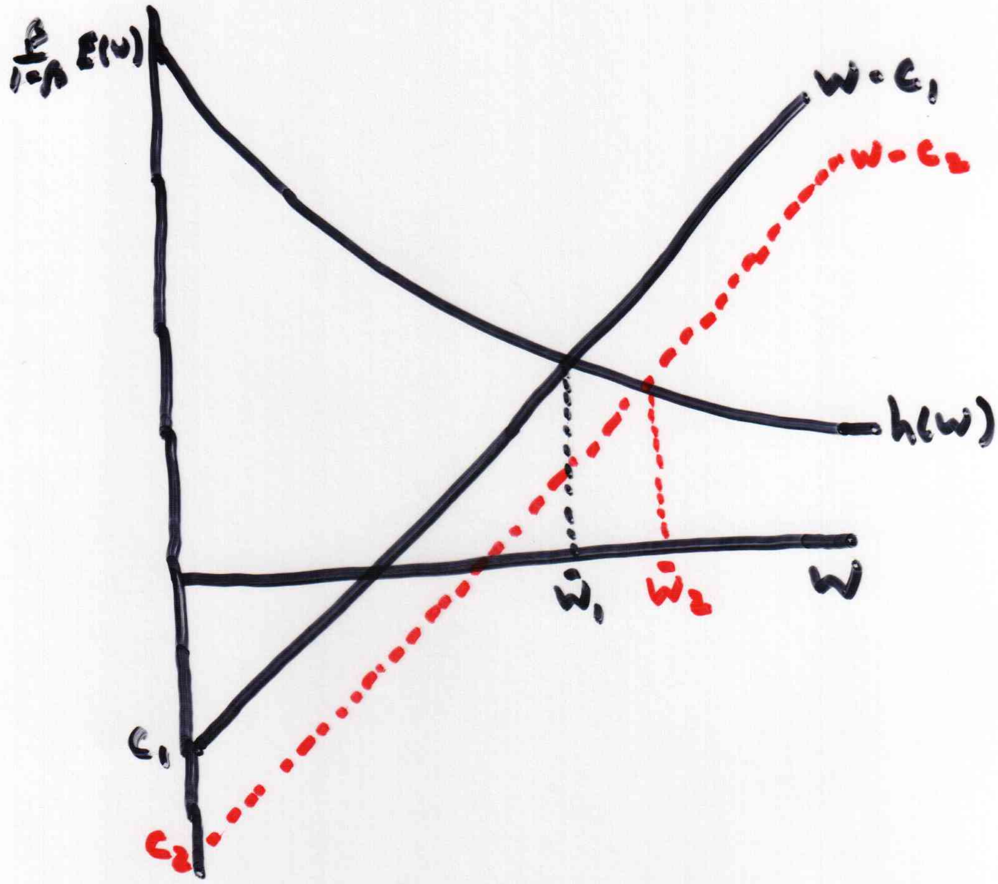
Leibniz Rule

$$\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

$$\phi'(t) = f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x,t)\,dx$$

We then also have $h''(w) = \frac{\beta}{1-\beta}F'(w) > 0$

- The following graph therefore characterizes the determination of the reservation wage, $\bar{w}$:



- Not surprisingly, $c \uparrow \Longrightarrow \bar{w} \uparrow$
- We could have also pursued an algebraic approach, by differentiating the optimality condition, viewing $\bar{w}$ as an implicit function of c

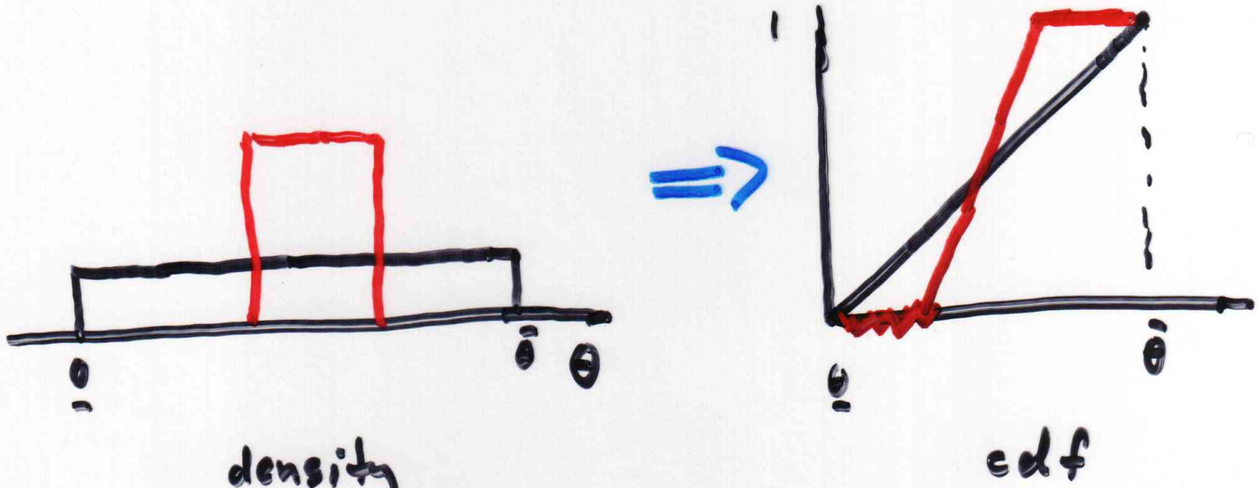
$$\frac{d\bar{w}}{dc} = 1 + \frac{\beta}{1-\beta}\int_{\bar{w}}^{B} -\left(\frac{d\bar{w}}{dc}\right) dF(w')$$

$$\Longrightarrow \frac{d\bar{w}}{dc}\left[1 + \frac{\beta}{1-\beta}(1-F(\bar{w}))\right] = 1$$

$$\Longrightarrow \boxed{\frac{d\bar{w}}{dc} = \left[1 + \frac{\beta}{1-\beta}(1-F(\bar{w}))\right]^{-1} > 0}$$

2.) Mean-Preserving Spread of the Wage Distribution

Consider the relationship between the following two densities and c.d.f's

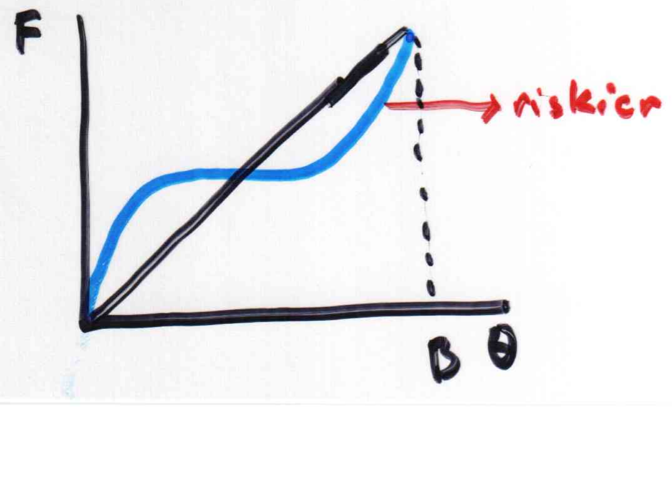


Note, the "riskier" cdf lies above the "safer" one for low θ (with the opposite being true for high θ).
This generalizes, and leads to the following:

Let $F(\theta, r)$ be a parameterized distribution. A **mean preserving spread** satisfies the following 2 conditions:

1.) $\int_0^B F_r(\theta, r)\,d\theta = 0$

2.) $\int_0^x F_r(\theta, r)\,d\theta \geq 0 \quad \forall\ 0 \leq x \leq B$



- To use this, let's rewrite the optimality condition,

$$\bar{w} - c = \frac{\beta}{1-\beta}\int_{\bar{w}}^{B}(w' - \bar{w})\,dF(w')$$

$$= \frac{\beta}{1-\beta}\int_{\bar{w}}^{B}(w' - \bar{w})\,dF(w') + \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})\,dF - \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})\,dF$$

$$= \frac{\beta}{1-\beta}\int_{0}^{B} w'\,dF - \frac{\beta}{1-\beta}\bar{w}\int_{0}^{B} dF - \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})\,dF$$

$$= \frac{\beta}{1-\beta}E(w) - \frac{\beta}{1-\beta}\bar{w} - \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})\,dF$$

$$(1-\beta)(\bar{w} - c) = \beta E(w) - \beta\bar{w} - \beta\int_{0}^{\bar{w}}(w' - \bar{w})\,dF$$

$$\bar{w} - (1-\beta)c = \beta E(w) - \beta\int_{0}^{\bar{w}}(w' - \bar{w})\,dF$$

Integrate the last term by parts,

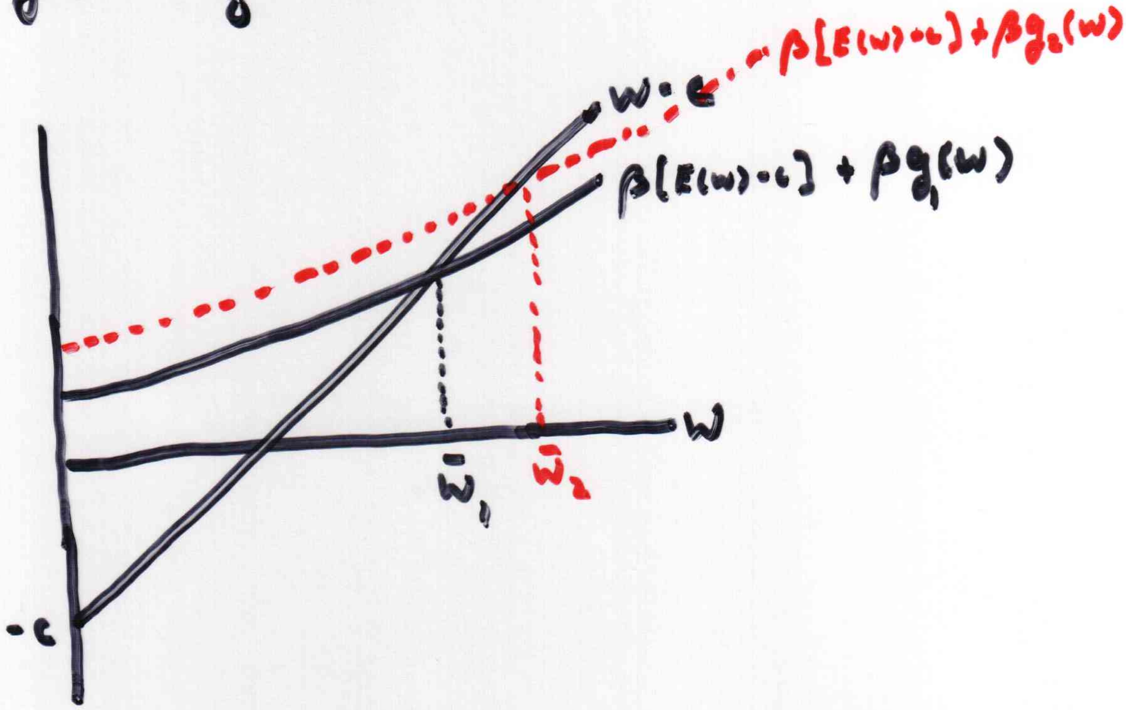
$$\int_{0}^{\bar{w}}\underbrace{(w' - \bar{w})}_{u}\,\underbrace{dF}_{dv} = \underbrace{(w' - \bar{w})F(w)\Big|_{0}^{\bar{w}}}_{=0} - \int_{0}^{\bar{w}}F(w')\,dw'$$

$$= -\int_{0}^{\bar{w}}F(w')\,dw'$$

Therefore, we get an alternative expression for the optimality condition,

$$\boxed{\bar{w} - c = \beta\left[E(w) - c\right] + \beta g(\bar{w})}$$

where $g(s) = \int_0^s F(x)dx$. We can graph this,



- By definition, a mean-preserving spread increases g, which produces a higher reservation wage.

Why?