Econ 815

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Book : "Asset Pricing" by John Cochrane (recommended, not required)

Grades :

Problem Sets	20%
Midterm	40%
Final	40%

9 Papers

1.) **Arrow** (1964) — Dynamic Spanning Contingent Claims
"The Role of Securities in the Optimal Allocation of Risk-Bearing"

2.) **Sharpe** (1964) — CAPM
"Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk"

3.) **Merton** (1971) — Dynamic Consumption / Portfolio Rules
"Optimum Consumption + Portfolio Rules in a Continuous-Time Model"

4.) **Black + Scholes** (1973) — Options Pricing
"The Pricing of Options + Corporate Liabilities"

5.) **Lucas** (1978) — General Equilibrium
"Asset Prices in an Exchange Economy"

6.) **Harrison + Kreps** (1978) — Heterogeneous Beliefs Bubbles
"Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations"

7.) **Harrison + Kreps** (1979) — Risk-Neutral Pricing
"Martingales + Arbitrage in Multiperiod Securities Markets"

8.) **Grossman + Stiglitz** (1980) — Informational Efficiency
"On the Impossibility of Informationally Efficient Markets"

9.) **Tirole** (1982) — No-Trade Theorems
"On the Possibility of Speculation Under Rational Expectations"

Omitted Topics

1.) Practical Implications
(Positive vs. Normative)

2.) Incomplete Markets + Financial Frictions

3.) Behavioral Finance

4.) Non-Expected Utility Models (e.g., Ambiguity)

5.) Market Microstructure

6.) Empirical Estimation + Evaluation !
(take Econ 818!)

Stochastic Processes

- We will spend a lot of time talking about stochastic processes.
- Asset prices, asset payoffs, investor wealth, and portfolio strategies can all be viewed as stochastic processes.
- Definition: A stochastic process is a time-indexed sequence of random variables.

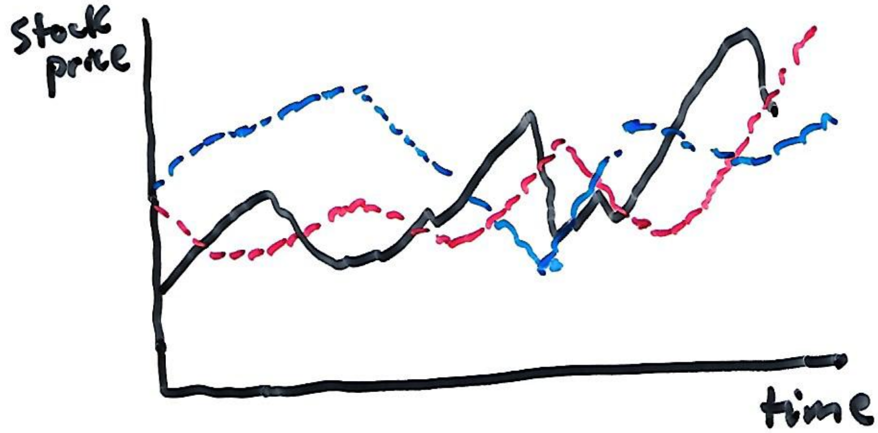
Stochastic Processes (paths) vs. Random Variables (real numbers)

- We need to regard a stochastic process as a single unit
- A stochastic process is a random variable, but it belongs to a different "space", i.e., a function space.

Comments

1.) A realization is an entire path, not a number.

2.) Population = Set of all possible realizations.



3.) We observe only one realization (unless you are a believer in reincarnation).

4.) "Law of Large Numbers" and "Central Limit Theorem" arguments still applicable if observations over time provide enough new information about ensemble averages.

Stationary Stochastic Process

A stochastic process where all joint distributions are independent of time. (Initial conditions have worn off).

$$f(x_t, x_{t+1}, \cdots x_{t+j}) \quad \text{indpt. of } t \ \forall j$$

Markov Process

A restriction on the conditional distribution function

$$f(\text{future} \mid \text{present, past}) = f(\text{future} \mid \text{present})$$ } No "path dependence"

or, $\text{Prob}(x_{t+1} \mid x_t, x_{t-1}, \cdots x_{t-k}) = \text{Prob}(x_{t+1} \mid x_t) \quad \forall k \geq 1$

- Whether a process is Markov depends on the dimensionality of the state.

Example: $S_t = \alpha_1 S_{t-1} + \alpha_2 S_{t-1} + \varepsilon_t$ appears non-Markov. However, if we define $Z_t = (S_t, S_{t-1})$ then Z_t is Markov.

- Note; A process can be Markov but non-stationary

    Example: $x_t = \lambda x_{t-1} + \varepsilon_t$ with $|\lambda| > 1$ (Why?)

- If a process is stationary and (low-dimensional) Markov, then "conventional" statistical methods can be applied.

Continuous- vs. Discrete-Time Stochastic Processes

- The time index in a stochastic process can either be discrete (integers), or continuous (real numbers).
- Is time "really" discrete or continuous ?
  [Who knows, ask a physicist].
- In econ + finance the choice between the two is based solely on mathematical + computational convenience. Use whichever is easier for the problem at hand !
- The fact is, in many asset pricing problems, continuous time is easier. This is especially true in option pricing.
- However, the best way to think about a continuous-time stochastic process is as a <u>limit</u> of a discrete-time process.
  [Note: This is not the most rigorous or mathematically sophisticated approach. A more rigorous approach would blast right off with function spaces + measure theory].

Wiener Process (Brownian Motion)

- Defining a discrete-time process is simple - Just cumulate i.i.d. shocks. The Wold Representation Theorem tells us that this is a completely general way to define a (linear) discrete-time process.

  Example: $x_t = \rho x_{t-1} + \varepsilon_t \quad |\rho|<1 \quad \varepsilon_t \sim \text{i.i.d}$

  $$\Rightarrow x_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}$$

- Likewise, defining a deterministic continuous-time process is also simple. Just write down a differential equation

  Example: $\frac{dx}{dt} = a \cdot x$

- Here's the question - Can we define a stochastic continuous-time process by just adding on an i.i.d shock, as we did in the discrete-time case above? That is, does the following make sense?

  $$\frac{dx}{dt} = ax + \varepsilon$$

  Answer: Not really, but in a way yes.

- It turns out that defining a continuous-time i.i.d. process is a bit tricky, and leads to some surprising results.

- We will see that the mathematically correct way to define a stochastic differential equation is as the following integral equation:

  $$x_t = x_0 + \int_0^t a x_s \, ds + \int_0^t dW_s$$

  where W_t is a Wiener process, and the 2nd integral is a new type of integral, called an Ito integral.

- The Wiener process can be viewed as a continuous-time limit of the following random-walk process:

$$X_t = X_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim \text{iid } N(0,1)$$

[Note: This is nonstationary. Why?]

$$\Rightarrow X_{t+T} = X_t + \sum_{j=1}^{T} \varepsilon_{t+j}$$

where the step-size shrinks to zero in a very particular way as the time between steps shrinks to zero.

- Note 3 features of the above random walk:

  1.) $E(X_{t+T} \mid X_t) = X_t \quad \forall T$

  or $E(\Delta X_{t+k} = 0 \mid X_t)$

  [Markov, with indpt., zero mean increments]

  2.) $Var(X_{t+T} \mid X_t) = T$

  [variance increases linearly with the number of periods]

  3.) Although the unconditional distribution is undefined (since the process is nonstationary), all conditional distributions are Gaussian.

- We want to derive a continuous-time process with these same properties. In particular, we want the variance to depend on the sample length, T, but not on the length or number of steps.

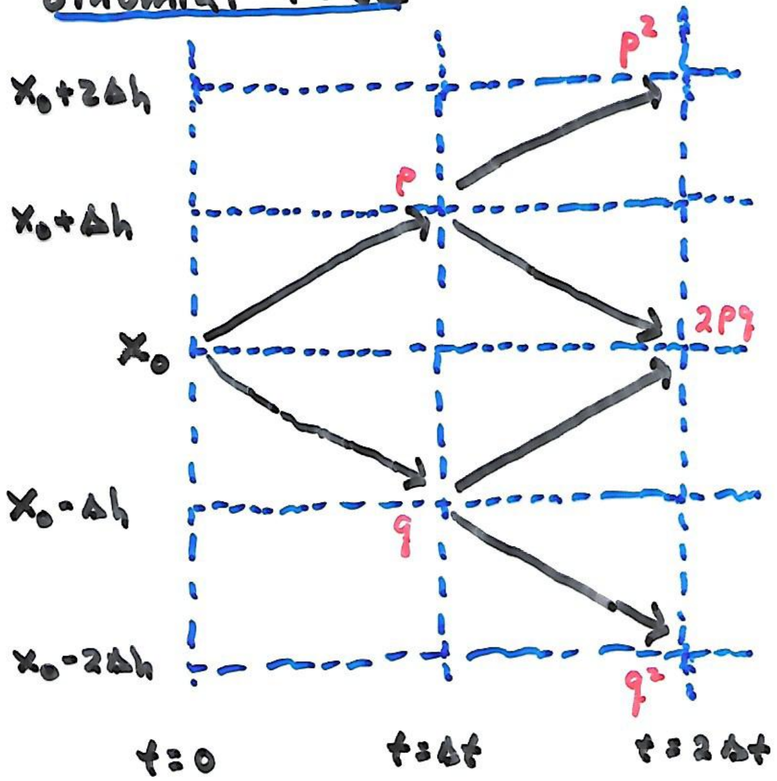
- Unless the step-sizes go to zero slower than the step-lengths, then the law of large numbers would cause the variance to shrink to zero.

- In analogy to the Wold Rep. Theorem, all continuous sample path continuous-time processes can be constructed as functions of the Wiener process. [If you want to allow jumps, then you must consider a wider class of innovations, called Levy Processes.

Binomial Tree

- Let's consider a slight generalization of a random walk, which allows the process to drift up ~~or~~ down on average over time. This will be important when studying asset prices.

- To visualize the path of this process consider the following binomial tree



Δh: step-size

Δt = step-length

- This depicts the path of the following r.w. (with drift) process

$$X_{t+\Delta t} = X_t + \varepsilon_{t+\Delta t} \qquad \varepsilon_t = \Delta h \text{ with prob. } p$$
$$= -\Delta h \text{ with prob. } q$$

- It is called a binomial tree because each step consists of the outcome of a Bernoulli 'trial', so the path of process (sum of the steps) is a binomial r.v.

- By definition, $\Delta t \to 0$ when defining a cont.-time process.
- The question is, what happens to Δh as $\Delta t \to 0$?
- Clearly, if the process is to be continuous, then $\Delta h \to 0$ as $\Delta t \to 0$. But how fast?

Moments

$$E(\Delta X) = (p-q)\Delta h$$

$$E[(\Delta X)^2] = p(\Delta h)^2 + q(-\Delta h)^2 = (\Delta h)^2$$

$$\mathrm{var}(\Delta X) = E[(\Delta X)^2] - [E(\Delta X)]^2 = [1-(p-q)^2](\Delta h)^2 = 4p\cdot q(\Delta h)^2$$

- A continuous-time interval of length T can be divided into $n = \frac{T}{\Delta t}$ discrete time-steps. Since each step is independent, we have

$$E(X_T - X_0) = \frac{T}{\Delta t}(p-q)\Delta h = T(p-q)\frac{\Delta h}{\Delta t} \qquad (1)$$

$$\mathrm{var}(X_T - X_0) = \frac{T}{\Delta t}\cdot 4pq(\Delta h)^2 = T\cdot 4pq\frac{(\Delta h)^2}{\Delta t} \qquad (2)$$

- Suppose $\Delta h \sim \Delta t$ (Δh goes to 0 at the same rate as Δt). Then notice from (2) that $\mathrm{var}(X_T - X_0) \to 0$. This is just the Law of Law Numbers. However, we want $\mathrm{var}(X_T - X_0) \sim T$
- To get this, we assume $\boxed{\Delta h = \sigma\sqrt{\Delta t}}$ [σ is just a scale factor]

This implies $\frac{(\Delta h)^2}{\Delta t} \to \sigma^2$

- If we also let

$$p = \frac{1}{2}\left[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right] \qquad q = \frac{1}{2}\left[1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$$

then, as $\Delta t \to 0$, the above binomial dist. converges to a Normal dist (i.e., Central Limit Theorem) and we find,

$$X_T - X_0 \longrightarrow N(\alpha T, \sigma^2 T)$$

[Note
$pq = \frac{1}{4}\left(1 - \frac{\alpha^2}{\sigma^2}\Delta t\right) \to \frac{1}{4}$]

- This is what we want!

Comments

- Notice that,

$$\frac{\Delta X}{\Delta t} \sim \frac{\Delta h}{\Delta t} \sim \frac{\sqrt{\Delta t}}{\Delta t} \sim \frac{1}{\sqrt{\Delta t}} \longrightarrow \infty \quad \text{as } \Delta t \longrightarrow 0$$

- This means that the continuous-time limit process, $X(t)$, is <u>not</u> differentiable.
- However, by construction, it <u>is</u> continuous.
- Can you visualize a continuous, but nowhere differentiable process (without drugs)? I can't.