Topics for Today

1.) Comment on CRRA vs. CARA in the Merton model

2.) Labor Income

3.) Learning

4.) Midterm Review

Comment on CRRA vs. CARA

- Last time we solved the Merton problem with CRRA preferences:

$$\left.\boxed{\begin{aligned} c &= \tfrac{1}{\gamma}\left\{\rho - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right\} W \\ \alpha &= \frac{\mu-r}{\gamma\sigma^2}\end{aligned}}\ \right\} \text{CRRA}$$

- With CRRA, a rich person is just a "scaled-up" version of a poor person. Portfolio allocations are constant + identical, and consumption is a fixed fraction of wealth. This is caused by the interaction of multiplicative risk and CRRA.
- With CARA $\left[u(c) = -\frac{1}{\eta} e^{-\eta c}\right]$ we get

$$\left.\boxed{\begin{aligned} c &= rW + \left[\frac{\rho - r + (\mu-r)^2/2\sigma^2}{\eta r}\right] \\ \alpha &= \frac{\mu-r}{r\eta\sigma^2 W}\end{aligned}}\ \right\} \text{CARA}$$

- Notice the agent now invests a fixed dollar amount in the risky asset. His share of wealth in the risky asset declines as his wealth grows. [Because his relative risk aversion increases].
- This portfolio behavior is grossly at odds with the data. That's why people prefer CRRA to CARA.
- However, notice that with CARA, rich people save a higher fraction of their wealth, which is arguably more consistent with the data.
- CARA is often used when shocks are additive, since there are no wealth effects in this case. Idiosyncratic labor income risk is an example.

Labor Income

- The basic Merton model assumes all income derives from invested wealth. This may be a decent approximation for Donald Trump, but for the rest of us, labor income is very important.

- In principle, it is straightforward to incorporate labor income. Let y = labor income, and assume it follows the process:

$$\boxed{dy = \lambda(y)\cdot dt + \sigma_y\, dB} \quad \}\text{ labor income}$$

where B = Brownian Motion. The budget constraint becomes:

$$\boxed{dw = \left[(r+\alpha(\mu-r))W + \lambda(y) - c\right]dt + \alpha^2\sigma^2W^2\cdot dz + \sigma_y\, dB}$$

- Now y becomes a state variable, and so the HJB equation contains a few additional terms,

$$V_y\cdot\lambda(y) + \tfrac{1}{2}\sigma_y^2\cdot V_{yy} + \sigma_y\cdot\sigma\,\rho\, V_{wy}$$

where ρ = correlation between y and risky asset.

- Note, finance researchers typically ignore the disutility of labor, and simply assume labor income is exogenous. However, with endogenous labor supply, agents can adjust their labor in response to market outcomes (e.g., delay retirement if the market crashes).

- In general, even with exogenous labor, the problem becomes unsolvable (at least analytically). However, there are a couple interesting cases to consider.

Case 1: Labor income is risk-free and tradeable. This case is discussed in section 7 of Merton. In this case, the agent can fully "capitalize" his labor income and incorporate it into his initial financial wealth. For example, if $y = \bar{y}$ is constant, then his "human capital" is $H = \bar{y}/r$, and this just gets added to initial financial wealth. The problem then becomes identical to the original one.

Case 2: Labor income is risk-free and non-tradeable. In practice, people cannot (fully) borrow against their future labor income, even if it's risk-free, due to obvious 'moral hazard' considerations. In this case, the agent effectively has a risk-free investment in the form of labor income. He can then simply adjust his financial portfolio to account for this. For example, let H = Human wealth and W = Financial wealth. Then invest $\alpha(W+H)$ dollars in the risky asset, and $(1-\alpha)(W+H) - H$ in the risk-free. Notice that the share of financial wealth invested in the risky is

$$\hat{\alpha} = \frac{\alpha(W+H)}{W} = \alpha(1 + H/W)$$

Implication: H/W changes over the lifecycle. It is high when you are young (both because H is high and W is low), and low when you are old. Hence, young people should hold more equities in their financial portfolios relative to old people. This perhaps explains a very common piece of investment advice.

Learning

- Both Sharpe + Merton assume investors know the mean return, μ, on the market. In practice, investors must estimate this using historical data.

- Estimates of μ vary over time as returns are realized. In general, this estimation risk influences portfolio choice.

- Let (g_t, ν_t) be time-t mean + variance of the estimate of μ. They evolve according to Bayes Rule (or 'Kalman Filter'):

$$\left.\begin{aligned} dg_t &= \frac{\nu_t}{\sigma^2}\left(\frac{dP_t}{P_t} - g_t \cdot dt\right) \\ d\nu_t &= \frac{-\nu_t^2}{\sigma^2} \cdot dt \end{aligned}\right\} \text{Kalman Filter}$$

- Now (g_t, ν_t) become state variables in the HJB equation.

- Since μ is unobserved, we replace it with g_t by transforming the shock process:

$$\frac{dP_t}{P_t} = g_t \cdot dt + \sigma \, d\tilde{Z} \qquad d\tilde{Z} = \frac{1}{\sigma}\underbrace{\left(\frac{dP_t}{P_t} - g_t \cdot dt\right)}_{\text{forecast error}}$$

"innovation"

- The value function now takes the form:

$$V(g, v, w) = H(g, v) \frac{w^{1-\gamma}}{1-\gamma}$$

and the portfolio rule becomes:

$$\alpha_t = \frac{g_t - r}{\gamma\sigma^2} + \underbrace{\frac{1}{\gamma} \frac{H_g}{H} \frac{v_t}{\sigma^2}}_{\text{Hedge}}$$

- Note, the portfolio share varies over time now.

- The second component is called a 'hedge'. One can show $H_g < 0$ when $\gamma > 1$. Hence, if investors are sufficiently risk averse, estimation risk causes them to reduce their demand for risky assets. Learning makes assets riskier because it creates a positive correlation between realized returns and estimates of their mean return.