Topics for Today

1.) The Black-Scholes Formula

- Assumptions + Notation
- Replicating Portfolio
- The No Arbitrage PDE
- Conversion to the 'Heat Equation'
- Using Fourier Transforms to Solve the Heat Equation
- Alternative Derivation I (Delta Hedging + "The Greeks")
- Alternative Derivation II (Risk-Neutral Pricing + Monte Carlo Simulation)
- Extensions + Complications
- Exotic Options
- Empirical Evaluation / Implied Volatility

Assumptions & Notation

1.) Underlying stock price governed by Geometric Brownian motion

2.) Riskless interest rate is constant

3.) Continuous-trading with no transactions costs

4.) No borrowing constraints or limits on short-selling

5.) Stock does not pay dividends (or produce other cash flows)

6.) No arbitrage possibilities!

Notation

S : Stock Price $\left(\frac{dS}{S} = \mu\, dt + \sigma\, dz \right)$

r : riskless rate $\left(\frac{dB}{B} = r\, dt \right)$

K : Strike Price

T : Expiration date

$C(S,t)$ = Price of call option

Replicating Portfolio

- Last time we derived a 1-period option pricing formula using a simple binomial tree argument. Now we extend to continuous-time.

- By Ito's Lemma we have,

$$dC = \frac{\partial C}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt + \frac{\partial C}{\partial t} dt$$

$$= \left(\frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + \frac{\partial C}{\partial t}\right)dt + \sigma S \frac{\partial C}{\partial S} \cdot dz$$

- Form a portfolio, P, consisting of x \$ in stock and y \$ in bonds, so that $P = x \cdot S + y \cdot B$. Therefore,

$$dP = x\,dS + y \cdot dB$$

$$= x[\mu S\,dt + \sigma S\,dz] + y r B\,dt$$

$$= (x\mu S + y r B)\,dt + x\sigma S \cdot dz$$

- Now pick x and y to replicate the option contract. Matching the dz terms we get

$$\boxed{x = \frac{\partial C}{\partial S}}$$

Next, pick y so that $P = \frac{\partial C}{\partial S} \cdot S + yB = C$. This gives,

$$\boxed{y = \frac{1}{B}\left(C - \frac{\partial C}{\partial S} \cdot S\right)}$$

The No Arbitrage PDE

- Plug x + y into the previous expression for dP

$$dP = (x\mu S + y r B)\,dt + x\sigma S \cdot dz$$
$$= \left[\frac{\partial c}{\partial S}\mu S + r\left(c - \frac{\partial c}{\partial S}\cdot S\right)\right]dt + \frac{\partial c}{\partial S}\sigma S \cdot dz$$

- Note that the dz term is the same as the dz term in dc. Matching the drift (dt) terms gives:

$$\frac{\partial c}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \frac{\partial c}{\partial t} = \frac{\partial c}{\partial S}\mu S + r\left(c - \frac{\partial c}{\partial S}\cdot S\right)$$

- Re-arranging we get,

$$\boxed{rC = \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}}$$

> Black-Scholes PDE!

- This must be solved subject to the boundary conditions:

$$\boxed{\begin{aligned} c(S,T) &= \max\{S_T - K, 0\} \\ c(0,t) &= 0 \end{aligned}}$$

Conversion to the Heat Equation

- In general, PDEs do not have analytical solutions. However, it turns out that with appropriate transformations, the Black-Scholes PDE can be converted to the "Heat Equation", which is a famous PDE in mathematical physics with a well known solution.
- To see this, define the following change of variables:

$$\boxed{x = \log(S/K)} \qquad \boxed{\tau = \tfrac{1}{2}\sigma^2(T-t)}$$

and define $C(S,t) = K \cdot V(x,\tau)$

- Note (using the chain rule)

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 K \frac{\partial V}{\partial \tau}$$

$$\frac{\partial C}{\partial S} = \frac{K}{S}\frac{\partial V}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{K}{S^2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right)$$

- This implies the following equation for $V(x,\tau)$

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV$$

where $\boxed{k = \frac{2r}{\sigma^2}}$

- Note that the boundary condition at $t = T$ becomes an <u>initial</u> condition at $\tau = 0$

$$c(S,T) = \max(S-K, 0) = \max(Ke^{x} - K, 0)$$

Since $v = \frac{1}{K} C \Rightarrow \boxed{v(x,0) = \max(e^{x}-1, 0)}$ > New initial condition

- Next, let's change variables again, and define,

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

$$\Rightarrow \quad v_\tau = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_\tau$$

$$v_x = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_x$$

$$v_{xx} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} u_x + e^{\alpha x + \beta \tau} u_{xx}$$

- This gives the following equation for u

$$u_\tau = u_{xx} + [2\alpha + (k-1)] u_x + [\alpha^2 + (k-1)\alpha - k - \beta] u$$

Note that if we choose $\alpha = -\frac{1}{2}(k-1)$ $\beta = -\frac{1}{4}(k+1)^2$

we get

$$\boxed{u_\tau = u_{xx}}$$

> Heat Equation

with transformed initial condition

$$u(x,0) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0 \right]$$

Using Fourier Transforms to Solve the Heat Equation

- The basic idea behind all transform methods is the following:
  1.) Use the transform to convert a hard eq. into an easy eq.
  2.) Solve the easy equation
  3.) Invert the transform

- Given a function, $f(x)$, the Fourier transform and its inverse are:

$$\mathcal{F}[f] = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega$$

Fourier Transform pair

- Note the following

$$\mathcal{F}[f_x] = i\omega\,\mathcal{F}[f]$$

$$\mathcal{F}[f_{xx}] = -\omega^2\,\mathcal{F}[f]$$

$$\mathcal{F}[f_t] = \frac{\partial}{\partial t}\mathcal{F}[f]$$

$$\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g]$$

convolution

where $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)g(s)\,ds$

$$\Rightarrow \mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} = f*g$$

convolution Property

• Letting $\hat{u} = \mathcal{F}[u]$ we then get

$$\frac{d\hat{u}}{d\tau} = -w^2 \hat{u}$$

} Fourier Transform of the Heat Equation

Note, our PDE is now an ODE! The solution is:

$$\hat{u} = \hat{u}(0)\, e^{-w^2 \tau}$$

where $\hat{u}(0) = \mathcal{F}[u(x,0)]$.

• Now invert using the Convolution Property,

$$u(x,\tau) = \mathcal{F}^{-1}[\hat{u}] = \mathcal{F}^{-1}[\hat{u}(0)] * \mathcal{F}^{-1}[e^{-w^2\tau}]$$

• This implies,

$$\boxed{u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} u(s,0)\, e^{-\frac{(x-s)^2}{4\tau}}\, ds}$$

> Solution of the Heat Eq.

Comment

1.) The function $e^{-\frac{(x-s)^2}{4\tau}}$ goes by various names. Physicists call it a 'heat Kernel'. Mathematicians call it a "Green's function". Economists refer to it as an "impulse response function".

The Black-Scholes Formula

- To finally get the Black-Scholes formula we just need to unwind all the transformations we did. This gives

$$C(S,t) = S \cdot \Phi\left[\frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right]$$

$$- K e^{-r(T-t)} \, \Phi\left[\frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right]$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds$ is the Normal CDF.

- This is usually written as:

$$\boxed{C(S,t) = S \cdot \Phi(d_1) - K e^{-r(T-t)} \cdot \Phi(d_2)}$$

Black Scholes Formula!

where $d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$

$d_2 = d_1 - \sigma\sqrt{T-t}$

- Note $(d_1, d_2) \to +\infty$ as $t \to T$ if $S > K$ and $\to -\infty$ if $S < K$, which implies

$$C(S,T) = \max\{S - K, 0\}$$ check ✓

Alternative Derivation I

- Black + Scholes actually derive their PDE using a different (but equivalent) portfolio strategy.

- Rather than form a portfolio of the stock + bond which replicates the option contract, they form a portfolio consisting of the stock + option which is riskless.

- As before, let $C(S,t)$ be the value of the option, and let $P = \theta_1 C(S,t) + \theta_2 S$ be the value of a portfolio consisting of θ_1 unit of the option and θ_2 shares of stock.

- Again, from Ito's Lemma,

$$dP = \theta_1 \left[C_t + C_S \cdot dS + \tfrac{1}{2}\sigma^2 S^2 C_{SS} \right] dt + \theta_2 \, dS$$

- Suppose we set $\theta_1 = 1$ and $\theta_2 = -C_S$. Note that the terms involving dS cancel, and we get

$$dP = \left(C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS} \right) dt$$

- Note, this is non-random and so is riskless. Therefore to avoid arbitrage we must have

$$dP = rP\,dt = r\left(C - S \cdot C_S\right)dt$$
$$= C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS}$$

$$\Longrightarrow \boxed{rC = C_t + rSC_S + \tfrac{1}{2}\sigma^2 S^2 C_{SS}}$$

This is the same PDE as before!

Delta Hedging and "The Greeks"

- The idea of going long one call & short $\frac{\partial c}{\partial S}$ shares of stock to form a riskless portfolio is called "delta hedging", where $\Delta = \frac{\partial c}{\partial S}$.

- Note, we could equivalently go long Δ shares of stock and short one call.

- More generally, the sensitivities of $c(S,t)$ to market conditions can be useful for hedging purposes.

- We have,

$$dc = \frac{\partial c}{\partial S} dS + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}(dS)^2 + \frac{\partial c}{\partial t} dt$$

$$= \underbrace{\Delta}_{\text{Delta}} \cdot dS + \frac{1}{2}\underbrace{\Gamma}_{\text{Gamma}} \cdot (dS)^2 + \underbrace{\Theta}_{\text{Theta}} \cdot dt$$

where (Δ, Γ, Θ) can be derived from the BS formula (e.g., $\Delta = N(d_1)$).

- Collectively, traders refer to these hedge ratios as "The Greeks". They tell you how the value of the option should change as a function of time and the underlying share price.