Topics for Today

1.) A few last comments on option pricing & the Black-Scholes formula

2.) The Lucas (1978) model / The "Consumption-Based CAPM"

- Economics vs. Finance (Summers (1985))
- Euler Equations & Risk Premia
- Market Efficiency $\not\Rightarrow$ Random Walks!

Alternative Derivation II

- Last time we derived the Black-Scholes PDE in 2 different ways. Solving this PDE subject to the appropriate boundary condition produces the Black-Scholes formula.

- From our previous analyses, you should have noticed a close connection between the BS PDE and a stationary HJB equation, where the drift in S (μ) is replaced by the riskless rate (r). This was no accident.

- The substitution of r is the essence of the "Risk-Neutral Approach" to derivatives pricing. This is a more "modern" approach. It is more flexible and easily applied.

- It exploits a deep mathematical connection between PDEs and the expected values of diffusion processes. We will explore this in more detail when we get to the Harrison & Kreps (1979) paper. For now I will just note that there is an equivalence between (certain) 2nd-order PDEs and the expected value of a diffusion process conditional on a boundary value.

- Expected values are easy to compute numerically, since they just involve taking averages. We can use the computer to generate many "random" realizations of S, and then simply compute the option price by computing an average.

- Here is what you do (for the Brownian Motion case):

  For $i = 1$ to SimNum

  1.) Divide the interval $[0,T]$ into steps of length Δt.

  2.) Generate $T/\Delta t$ draws from a $N(0, \Delta t)$ distribution

3.) Compute a sequence of stock prices

$$S(t_k + \Delta t) = S(t_k) + rS(t_k)\Delta t + \sigma S(t_k)\epsilon(t_k) \quad k = 1, 2, \ldots T/\Delta t$$

4.) Compute $f(S(T))$, where $f(\cdot)$ is the boundary condition. For a call it is $f(S(T)) = \max[S(T) - K, 0]$.

5.) At the end of the loop compute

$$\boxed{\hat{C} = e^{-rT} \text{average}[f(S(T))]}$$

Comments

1.) This Monte Carlo approach easily handles general processes for S [it does not have to be geometric Brownian Motion!]

2.) It easily handles complicated (even path dependent!) boundary conditions.

3.) However, it does not easily handle American-style options which may be exercised early.

4.) Also, it is not well suited to 'real-time' decision making and high-frequency trading.

Extensions and Complications

- In a sense, the BS formula is very robust, since it is based on simple no arbitrage logic. However, to derive analytic formulas one has to make assumptions. Many finance professors have spent their careers exploring modifications of the BS assumptions. Among the most important are:

1.) Time-Varying riskless rate

Punchline: If you are pricing a 6-month option, use the 6-month T-bill rate. If you are ~~using~~ pricing a 1-year option, use the 1-year T-bill rate, etc.,

2.) Time-Varying Volatility

If volatility varies as a known function of time, BS remains valid if you replace σ with its average value over the life of the option. Stochastic volatility produces more significant changes. If its uncorrelated with the asset price, BS remains valid with an appropriate averaging procedure. Correlated stochastic volatility produces major changes to BS.

3.) Dividends

In general, if a stock pays dividends you can no longer derive an analytic formula. However, in some special cases you can. For example, if dividend payments are a constant fraction of the stock price (constant 'dividend yield'), then a simple adjustment to the BS formula results. [Prob Set 2 asks you to explore this].

4.) Transaction Costs

All hell breaks loose. (You can no longer maintain a riskless portfolio!).

Exotic Options

- Exchange traded options are pretty standardized (calls, puts, + a few variants). However, in the OTC market, the sky is the limit, + there are a bewildering variety of exotic options:

1.) **Lookback Options**: The strike price is determined by the minimum value of the stock (or max for a put).

2.) **Asian Options**: The payoff depends on the average value of the stock. Example 1: $C_T = \max[S_T - S_{avg}, 0]$.
Example 2: $C_T = \max[S_{avg} - K, 0]$.

3.) **Knock-Out Options**: The option expires immediately if the asset price hits a barrier. A rebate is then paid.

4.) **Forward Start options**: The option is purchased at one date, but does not begin until a later date.

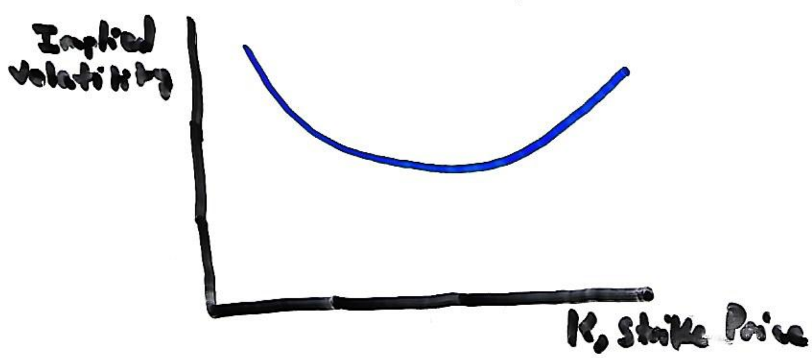
5.) **Chooser Options**: After a specified time, the holder can choose whether it's a call or a put.

6.) **Compound Options**: An option on an option.

Empirical Evaluation / Implied Volatility

- In principle, the BS formula should hold exactly, since it's based on no arbitrage.
- Although it works well, there are discrepancies.
- One possible reason is that the assumptions about transaction costs are invalid. However, the fact that it often does work, suggests this might not be the main issue.
- Note that all the BS parameters (S, K, T, r, σ) are observable, except one, namely σ.
- We can always make the BS formula fit perfectly if we pick σ appropriately. For a given option, this value is called the implied volatility.
- However, there are many options written on the same stock. They should all give the same implied volatility. In practice, they often differ. There is an intriguing pattern that is often generated, called the Volatility Smile:

Volatility Smile



- The volatility smile suggests that deep-in-the-money and deep-out-of-the-money options are over priced. The smile is most prevelent in currency options.

The Lucas (1978) Model

- Here's a re-cap of what we've done so far:

Arrow – Very general. Everything (implicitly) depends on state-contingent claims prices. These prices are determined as usual by "tastes + technology", but no specific results obtained.

Sharpe – Explains individual stock returns by their correlations with the market. Says nothing about what determines fluctuations in the market.

Merton – Studies optimal policies for an individual. Asset prices are exogenous.

Black-Scholes – Pricing of derivative (redundant) assets. Says nothing about the dynamics of the underlying asset price, which is assumed exogenous.

- In short, we've said nothing about why the stock market fluctuates!

- Summers (1985) notes that this highlights a key difference between economists + finance professors.

- Our next 2 papers finally begin to address this crucial question.

- Lucas (1978) develops a General Equilibrium model that attributes market fluctuations to fluctuations in macroeconomic fundamentals. Harrison + Kreps (1978) develop a model of heterogeneous beliefs, which shows that markets can fluctuate because of speculation.

Euler Equations + Risk Premia

- Consider an investor who can trade a stock + bond. He solves:

$$\max E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

s.t. $c_t + R_t^{-1} L_t + P_t N_t \leq A_t$

$$A_{t+1} = L_t + (P_{t+1} + y_{t+1}) N_t$$

L_t : bond holding (claim to 1 unit of consumption)

N_t : stock holdings

P_t : stock price

y_t : dividend

Euler Equations

L_t: $u'(c_t) = \beta R_t E_t u'(c_{t+1})$

N_t: $P_t u'(c_t) = \beta E_t [u'(c_{t+1})(y_{t+1} + P_{t+1})]$

"Transversality Conditions"

$$\lim_{k \to \infty} E_t \beta^k u'(c_{t+k}) R_{t+k}^{-1} L_{t+k} = 0$$

$$\lim_{k \to \infty} E_t \beta^k u'(c_{t+k}) P_{t+k} N_{t+k} = 0$$

- Note, these eqs. must hold for <u>all</u> investors, regardless of whether markets are complete or incomplete. (They might be inequalities for some investors, who face borrowing or short-sale constraints.

- They impose <u>testable restrictions</u> on the relationship between asset prices + consumption. These restrictions are the same whether markets are complete or incomplete.