Topics for Today

1.) The Lucas Model

2.) Random Walks vs. Market Efficiency

3.) Examples

4.) A Log-linear Approximation

5.) The Equity Premium Puzzle

6.) Hansen-Jagannathan Bounds

7.) Strategies for Resolving the Equity Premium Puzzle

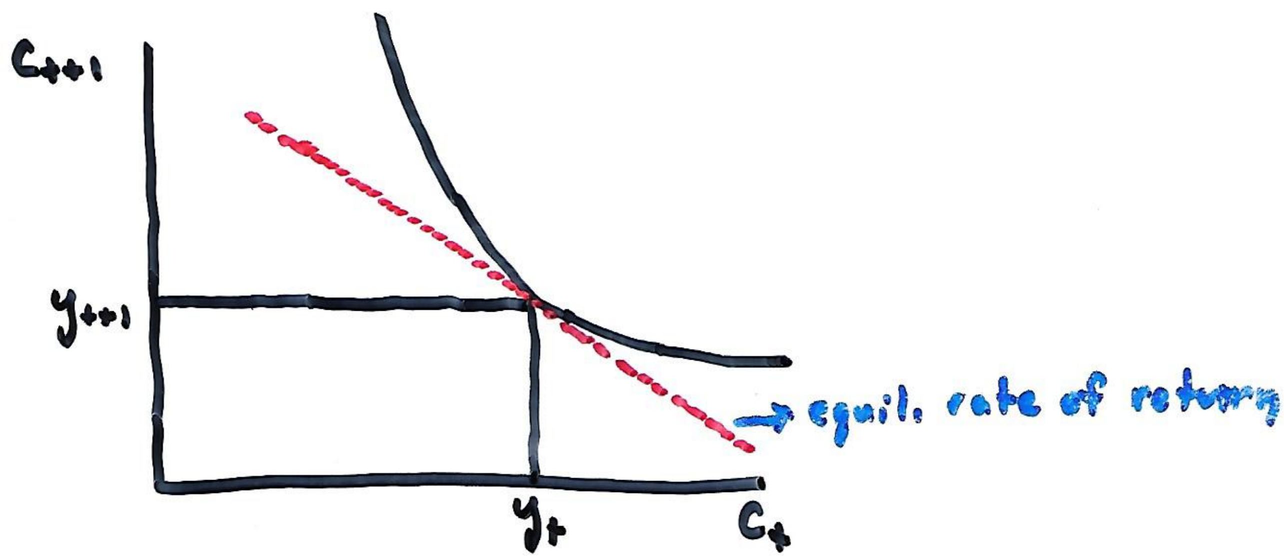
Lucas Model

Complete MKts. / Endowment Economy

Identical Preferences + Endowments

$\Longrightarrow$ In equil., everyone consumes (per capita) aggregate endowment, which consists of the stream of dividends from assets ("trees").

Graphically,



Comments

1.) Notice returns adjust so that everyone is content to consume their endowment. In equilibrium, there is no asset trade! These are "shadow prices". (Obviously, $L_t = 0$ in equil., since bonds are in zero net supply).

2.) Notice that Lucas just reverses the logic of Friedman's "Permanent Income Hypothesis". Friedman assumed asset returns were exogenous, and used the consumption Euler eq. to make statements about optimal consumption behavior. (These statements were formally tested by Hall '78). In contrast, Lucas assumes consumption is exogenous, + uses the Euler eq. to make statements about equil. asset prices.

That is,

Friedman/Hall PIH :	R_t exogenous / C_t endogenous
Lucas CCAPM Model :	R_t endogenous / C_t exogenous

- Of course, both are endogenous. However, the implied restrictions on the empirical relationship between C_t + R_t will be the same either way!

- One thing the Lucas model makes clear is that, in general, asset prices will not be random walks. (Contrary to popular belief).

To see this, consider the Euler Eq. →

From Euler,

$$P_t = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}(y_{t+1} + P_{t+1})\right]$$

$$= \beta E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) E_t(y_{t+1} + P_{t+1}) + \beta\, \mathrm{cov}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}, y_{t+1} + P_{t+1}\right]$$

Suppose investors are risk-neutral (u' is constant), then

$$P_t = \beta E_t(y_{t+1} + P_{t+1}) \implies$$ Prices follow a martingale, when adjusted for discounting + dividends

$$\implies P_t = E_t \sum_{j=1}^{\infty} \beta^j y_{t+j}$$ ↘ price = expected PDV of dividends

Comments

1.) Notice that, in general, asset prices in "efficient markets" are not random walks, due to potential variation in $E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)$ and/or $\mathrm{cov}_t(\cdot)$. This reflects "time-varying risk premia". Prices are martingales only when investors are risk neutral. Hence, predictable movements in asset returns do not (necessarily) reflect market inefficiency!

2.) Technically, a martingale only restricts the (conditional) first moment. Thus, it is less restrictive than a random walk, which restricts the entire distribution. For example, a martingale is consistent with ARCH/GARCH effects in the forecast errors. A random walk is not.

- In the risk-neutral case, we can easily solve for asset prices by evaluating the expected PDV of dividends (given an explicit process for dividends).

- More generally, however, it can be difficult to solve for equilibrium prices, due to potential time-variation in risk premia.

- I will first present 2 special cases where it **is** possible to obtain closed-form solutions. Then I will present a useful approximation for the general case.

Special Cases

(1.) **Log utility**: Imposing the equil. condition $c_t = y_t$ into Euler eq.,

$$P_t u'(y_t) = \beta E_t u'(y_{t+1})(y_{t+1} + P_{t+1})$$

$$= \cancel{\beta} E_t \sum_{j=1}^{\infty} \beta^j u'(y_{t+j}) y_{t+j}$$

Notice that with log util., $u'(y_{t+j}) = \frac{1}{y_{t+j}}$, so that

$$\boxed{P_t = \frac{\beta}{1-\beta} y_t}$$

Question: Why are the dynamics in y_t irrelevant?
Why is the P/D ratio constant?
(Hint: Think income + substitution effects).

(2.) **Discrete States:** Suppose dividends only occupy a finite # of states, $s_1, s_2, \ldots s_n$. Let

$$P_{ij} = \text{Prob}[y_{t+1} = s_j \mid y_t = s_i]$$

We then have the discrete-state Euler eq.

$$p(s_i)u'(s_i) = \beta \sum_{j=1}^{n} p(s_j)u'(s_j)P_{ij} + \beta \sum_{j=1}^{n} s_j u'(s_j)P_{ij}$$

Defining $v_i = p(s_i)u'(s_i)$ and $\alpha_i = \beta \sum_{j=1}^{n} s_j u'(s_j) P_{ij}$,

$$v_i = \alpha_i + \beta \sum_{j=1}^{n} P_{ij} v_j$$

or in vector-matrix notation,

$$\boxed{\begin{aligned} v &= \alpha + \beta P v \\ \Longrightarrow \quad v &= (I - \beta P)^{-1} \alpha \\ \Longrightarrow \quad p_i &= \frac{v_i}{u'(s_i)} \end{aligned}}$$

A Useful Approximation

Write the Euler eq. as follows,

$$1 = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{t+1}\right]$$

where $R_{t+1} = \frac{y_{t+1} + p_{t+1}}{p_t}$ = asset return (not price).

Next, apply "law of iterated expectations" to write in terms of unconditional expectations,

$$1 = E\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{t+1}\right]$$

Specialize to $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ (i.e., CRRA).

$$1 = E\left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}\right]$$ } a nonlinear function of 2 random variables.

Write this as,

$$1/\beta = E\left[(1+g_{t+1})^{-\gamma}(1+r_{t+1})\right]$$

where g_{t+1} = consumption growth rate

r_{t+1} = asset rate of return

Define the function, $f(g,r) = (1+g)^{-\gamma}(1+r)$ and consider the following 2nd-order Taylor series approx. (around $\bar{g}, \bar{r}$).

$$f(g,r) \approx f(\bar{g},\bar{r}) + \frac{\partial f}{\partial g}(g-\bar{g}) + \frac{\partial f}{\partial r}(r-\bar{r}) + \frac{1}{2}\frac{\partial f^2}{\partial g^2}(g-\bar{g})^2$$
$$+ \frac{1}{2}\frac{\partial f^2}{\partial r^2}(r-\bar{r})^2 + \frac{\partial f}{\partial g \partial r}(g-\bar{g})(r-\bar{r})$$

Approximating for small growth rates,

$$\bar{g} = \bar{r} = 0$$

$$f(\bar{g},\bar{r}) = 1$$

$$\frac{\partial f}{\partial g} = -\gamma \qquad \frac{\partial f}{\partial r} = 1$$

$$\frac{\partial^2 f}{\partial g^2} = \gamma(\gamma+1) \qquad \frac{\partial^2 f}{\partial r^2} = 0 \qquad \frac{\partial^2 f}{\partial g \partial r} = -\gamma$$

Sub in,

$$1/\beta \approx E\left[1 + r_{t+1} - \gamma g_{t+1} + \tfrac{1}{2}\gamma(\gamma+1)g_{t+1}^2 - \gamma g_{t+1} r_{t+1}\right]$$

Letting $1/\beta = 1 + \delta$

rate of time preference

$$1+\delta \approx 1 + E(r) - \gamma E(g) + \tfrac{1}{2}\gamma(\gamma+1)\left[var(g) + \underbrace{(Eg)^2}_{=0}\right] - \gamma\, cov(g,r) - \gamma E(g) \underbrace{E(g)}_{=0}$$

Eliminating the 2nd-order terms + solving for $E(r)$,

$$E(r) \approx \delta + \gamma E(g) - \tfrac{1}{2}\gamma(\gamma+1)\, var(g) + \gamma\, cov(g,r)$$

Defining $r_f \approx \underbrace{\delta}_{\text{impatience}} + \underbrace{\gamma E(g)}_{\text{intertemp. subst.}} - \underbrace{\tfrac{1}{2}\gamma(\gamma+1)\, var(g)}_{\text{precautionary saving}}$ } risk-free rate

We can then write,

$$\boxed{E(r) \approx r_f + \underbrace{\gamma}_{\text{price of risk}}\, \underbrace{cov(g,r)}_{\text{quantity of risk}}}$$ } CCAPM

Letting $\rho = corr(g,r)$, we can write this as,

$$\boxed{E(r) \approx r_f + \gamma \rho\, \sigma_g \sigma_r}$$ > Everything observable except γ

The Equity Premium Puzzle

- Let's apply this to the case of the U.S. stock market (other countries are similar), so that

  $E(r)$ = average real return (annual) on U.S. stock mkt.

  r_f = average real return on U.S. T-Bills.

$$E(r) - r_f = .07 - .01 = .06$$

$$\sigma_g = .01$$

$$\sigma_r = .16$$

$$\rho = .2$$

$$\Longrightarrow \quad \gamma = \frac{.06}{.2(.0016)} = 187!$$

(or $\gamma = 37$ if the theoretical value $\rho = 1$ is used)

Equity Premium Puzzle = Very high γ needed to explain observed equity premium

Sources: 1.) Consumption growth smooth relative to stock returns

2.) Consumption growth weakly correlated with stock returns.

Why not high γ?

1.) $\gamma \uparrow$ implies $r_f \uparrow$ (risk-free rate puzzle) unless γ is really big

2.) high γ contradicts observed (and common sense) choices about risk bearing

Hansen-Jagannathan Bounds

- Whenever an economic model is rejected by the data, an obvious response is to ask- "Maybe I have the wrong utility function".

Usual Strategy

1.) Specify a discount factor model (i.e., utility function)

2.) See if it works

Hansen + Jagannathan's Approach

1.) Derive the properties that <u>any</u> valid discount factor model must satisfy

2.) See if you can devise a theoretically consistent model that meets these requirements.

- That is, H+J pursue a nonparametric, reverse engineering approach.

- Start with, $P(x) = E(mX)$
- With a given utility function [that is date/state separable], we know $m = \beta \frac{u'(c_{t+1})}{u'(c_t)}$
- In what sense does this hold more generally?

1.) Law of One Price $\Longrightarrow$ Pricing functional is linear

mapping from payoff streams to real line

2.) Riesz Rep. Theorem $\Longrightarrow$ Linear functionals in a Hilbert Space have an innep product representation

Let $X \in$ Hilbert Space + $f(x)$ be a linear functional, then we can write $f(x) = y \cdot x$ for some vector y.

3.) Therefore, w.l.o.g, $P(x) = E(mX)$

inner product rep. for pricing functional

Other results,

a.) No arbitrage $\Longrightarrow$ $m > 0$

b.) Complete Mkts. $\Longrightarrow$ m is unique = AD state price density

- Note, m plays 2 important roles

$$\boxed{P(x) = E(mx) = \underbrace{E(m)}_{\text{time discount}} E(x) + \underbrace{\text{cov}(m, x)}_{\text{risk}}}$$

- Also note,

$$P(1) = E(m) = \text{Price of sure claim}$$
$$= \text{reciprocal of (gross) riskless interest rate}$$

Simple Example - Excess Returns

$$1 = E(mR)$$
$$1 = E(m \underbrace{R_f}_{\text{risk free rate}})$$
$$\Longrightarrow \quad 0 = E(mR^e)$$

where $R^e = R - R_f$ = "excess return"

Therefore,

$$E(m)E(R^e) + \text{cov}(m, R^e) = E(m)E(R^e) + \rho \sigma_m \sigma_{R^e}$$
$$= 0$$

Re-arranging,

$$\frac{E(R^e)}{\sigma(R^e)} = -\rho \frac{\sigma_m}{E(m)}$$

Since $|\rho| \leq 1$, we have

$$\boxed{\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma_m}{E(m)}}$$

} lower bound on discount factor volatility

Sharpe Ratio

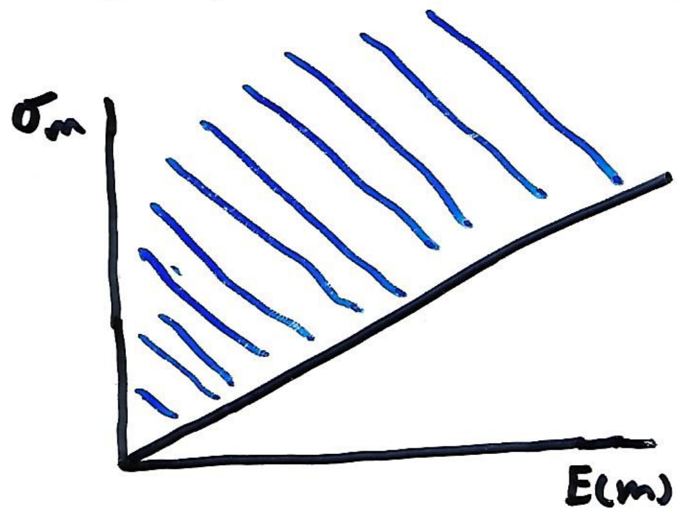
In U.S. Data, $E(R^e) \approx .06$, $\sigma(R^e) \approx .16$, thus

$$\frac{.06}{.16} = .375 < \frac{\sigma_m}{E(m)}$$

Multiple Assets: $\frac{\sigma_m}{E(m)} \geq \left[E(R^e)' \Sigma^{-1} E(R^e)\right]^{1/2}$

Σ = var-cov matrix of R^e

Graphically:



• Now extend to returns (rather than excess returns)

$$\underset{n\times 1}{1} = E(\underset{n\times 1}{R} \cdot \underset{1\times 1}{m})$$

• Note that $m^* = R'[E(RR')]^{-1} \cdot 1$ provides a valid discount factor for these given assets. Also note that $m = m^* + \varepsilon$, where ε is any random variable uncorrelated with R, also provides a valid discount factor.

Question: Which linear combo. of R delivers the minimum variance discount factor?

Define: $\mu_m = E(m)$ $\mu_R = E(R)$ $\Sigma = E(R-\mu_R)(R-\mu_R)'$

and project $m - \mu_m$ onto $R - \mu_R$

$$m - \mu_m = (R - \mu_R)'\beta + u$$

where $\hat{\beta} = \Sigma^{-1} E(R - \mu_R)(m - \mu_m)$ > Least squares Formula

From the pricing equation,

$$E(R-\mu_R)(m-\mu_m) = E(R\cdot m) - \mu_R\mu_m$$
$$= 1 - \mu_R\mu_m$$

$$\Longrightarrow \hat{\beta} = \Sigma^{-1}(1-\mu_R\mu_m)$$

By construction, u is orthogonal to regressors. Therefore,

$$\sigma_m^2 = E\left[\hat{\beta}'(R-\mu_R)(R-\mu_R)'\hat{\beta}\right] + \sigma_u^2$$
$$= (1-\mu_R\mu_m)'\Sigma^{-1}\Sigma\Sigma^{-1}(1-\mu_R\mu_m) + \sigma_u^2$$

Therefore,

$$\left.\boxed{\sigma_m^{\cancel{2}} \geq \sqrt{(1-\mu_R\mu_m)'\Sigma^{-1}(1-\mu_R\mu_m)}}\right\}\text{Hansen Jagannathan Bound}$$

- This defines a parabolic region in $(E(m), \sigma_m)$ space, which depends on the means and var-cov matrix of observed returns.
- Let's look at a couple examples

Hansen - Jagannathan Bound

U.S. Stock Mkt. + Treasury Bills

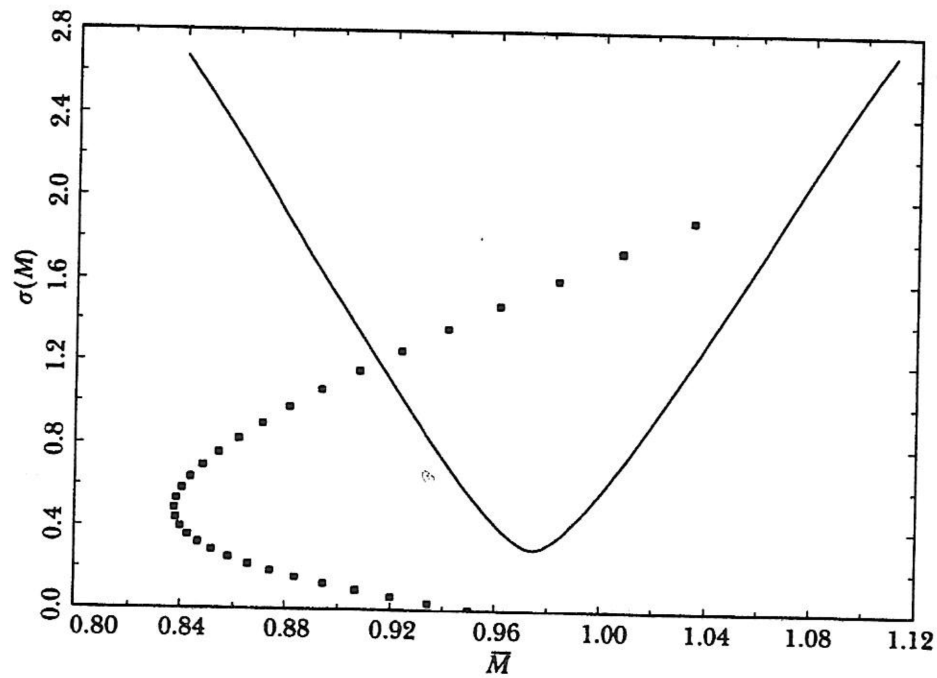


Figure 8.3. Feasible Region for Stochastic Discount Factors Implied by Annual US Data, 1891 to 1994

- Solid squares depict implied σ_m, $E(m)$ for traditional time + state separable CRRA preferences,

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

for $\beta = .98$ and $\gamma = 1, 2, 3, \ldots\ldots 31$

Strategies for Resolving the Equity Premium Puzzle

1.) Data / Measurement Biases

- Taxes (McGrattan + Prescott (AER, 2003))
- Survivorship Bias (Brown, Goetzmann + Ross (JF 1995))
- Rare Disasters / Peso Problems (Barro (QJE (2006))

2.) Exotic Preferences

- Date + State Nonseparabilities (Habit Persistence + Epstein-Zin)
- Ambiguity / Model Uncertainty

3.) Incomplete Markets

- Persistent Idiosyncratic Labor Income Risk (Mankiw (1986), Constantinides + Duffie (1996))