Topics for Today

1.) Information aggregation in Asset Markets

- Motivation
- Useful Facts and a Definition
- Grossman (1976)
  "Efficiency of Competitive Stock Markets Where Traders Have Diverse Information"

Motivation

- Last time we saw that heterogeneous beliefs generate trading, and can potentially generate asset price bubbles. Today we examine the role of asymmetric information.

- We start by studying the role that asset prices play in aggregating information.

- Normally, prices coordinate actions by constraining actions. When a good becomes scarce, its price increases, leading people to reduce their demand for the now scarcer good.

- In asset markets, prices play a dual role. They constrain choices as usual, but they also convey information. When an asset price increases, demand can actually increase, since investors might infer from the price increase that there is positive news about future payoffs. [Note: This effect can also occur for regular commodities if quality is unobserved, + price signals quality].

- Analyzing these types of models can be quite difficult. They confront investors (and their economic modelers!) with signal extraction problems.

- We start with some useful results about signal extraction problems.

Useful Facts and a Definition

- Signal extraction involves updating beliefs about an unobserved random variable based on observed signals/info. The fundamental result we need is the following:

Bayes Theorem: $P(A \mid B) = \dfrac{P(B \mid A)\, P(A)}{P(B)}$

- In general, this formula can be quite messy. But when the signals and underlying random variables are Normally distributed, it becomes very simple + intuitive.

- Let $x \sim N(\mu, \tau^{-1})$ be some underlying r.v. with mean μ, and variance τ^{-1}. [τ is called the precision].

- Suppose you observe a signal of x:

$$s = x + \varepsilon$$

where $\varepsilon \sim N(0, \alpha^{-1})$, where ε is uncorrelated with x.

- Applying Bayes Theorem we get:

$$\boxed{E(x \mid s) = \frac{\tau}{\tau+\alpha}\mu + \frac{\alpha}{\tau+\alpha} s} \qquad (1)$$

Hence, our revised estimate is just a weighted average of our prior estimate, μ, and our observed signal, s, with weights determined by relative precision. If the signal is more precise, it gets a larger weight.

- Again applying Bayes Theorem, the posterior variance becomes:

$$\boxed{\operatorname{var}(x \mid s) = (\tau + \alpha)^{-1}} \qquad (2.)$$

That is, the posterior variance is just the reciprocal of the sum of the prior precision and the precision of the signal.

- Another useful fact involves the expectation of an exponential function of a normally distributed random variable. Suppose $x \sim N(\mu, \sigma^2)$. Then we have:

$$\boxed{E\, e^{\gamma x} = e^{\gamma \mu + \frac{1}{2}\gamma^2 \sigma^2}} \qquad (3.)$$

- Formulas 1-3 are used over & over again in the literature on asset pricing with asymmetric information.

- In his paper, Grossman uses the following definition from statistics:

Sufficient Statistic

Let $(Y_1, Y_2, \cdots Y_n)$ be a random sample from $f(y; \theta)$, where θ is some parameter characterizing the density, $f(\cdot)$. Then $W = h(Y_1, Y_2, \cdots Y_n)$ is a sufficient statistic for θ iff the joint density of the Y_i's factors into the product

$$f(y_1, y_2, \cdots; \theta) = g_1(W(y_1, y_2, \cdots y_n); \theta) \times g_2(y_1, y_2, \cdots y_n)$$

In other words, all the relevant info about θ is contained in the function (or "statistic") $W(\cdot)$.

• Here's a famous example,

Let $Y_1, Y_2, \cdots Y_n$ be a sample from $N(\mu, \sigma^2)$, with σ^2 known.

Let $\bar{Y} = \frac{1}{n}\sum Y_i$. Note,

$$\sum (Y_i - \mu)^2 = \sum \left[(Y_i - \bar{Y}) + (\bar{Y} - \mu)\right]^2 = \sum (Y_i - \bar{Y})^2 + n(\bar{Y} - \mu)^2$$

because cross product sums to zero.

• Therefore, the joint p.d.f can be written,

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{\sum (Y_i - \mu)^2}{2\sigma^2}\right] = \exp\left[\frac{-n(\bar{Y} - \mu)^2}{2\sigma^2}\right] \times \left[\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{\sum (Y_i - \bar{Y})^2}{2\sigma^2}\right]\right]$$

Notice that μ only enters the first factor, which only depends on $\bar{Y}$.

Hence, $\bar{Y}$ is a sufficient statistic for μ.

Grossman (1976)

Assumptions

1.) Static, 2-period model. Invest in period 0. Consume wealth in period 1.

2.) Two assets: (i) risk-free, with return $(1+r)$, and (ii) Risky, which returns $\tilde{P}_1$ dollars in period 1.

3.) There are n (types of) informed traders, who behave competitively. Each gets (for free) a "piece of information" in the form of an unbiased signal of P_1 (the unknown true value of $\tilde{P}_1$):

$$y_i = P_1 + \varepsilon_i$$

where $\varepsilon_i \sim N(0, 1)$.

4.) Traders have a common prior, $N(\bar{P}_1, \sigma^2)$, about $\tilde{P}_1$.

5.) Traders have identical CARA preferences

$$U(\tilde{W}_{1i}) = e^{-a\tilde{W}_{1i}}$$

6.) Total supply of the risky asset is fixed, and is equal to $\bar{X}$.

Budget Constraints

$$W_{0i} = X_{Fi} + P_0 X_i$$

$$\tilde{W}_{1i} = (1+r) X_{Fi} + \tilde{P}_1 X_i$$

P_0 = market-clearing price in period 0

X_{Fi} = value of risk-free asset purchases

X_i = units of risky asset purchased

- Combing these two by eliminating X_{Fi},

$$\tilde{W}_{1i} = (1+r) W_{0i} + [\tilde{P}_1 - (1+r) P_0] X_i$$

- Using Fact 3 above,

$$E[u_i(\tilde{W}_{1i}) \mid I_i] = -\exp\{-[a E(\tilde{W}_{1i} \mid I_i) - \tfrac{a^2}{2} var(\tilde{W}_{1i} \mid I_i)]\}$$

where $I_i = (P_0, y_i)$ is trader-i's info set. Note that maximizing expected utility here is equivalent to the following mean-variance problem:

$$\boxed{\max_{X_i} E[\tilde{W}_{1i} \mid I_i] - \tfrac{a}{2} var[\tilde{W}_{1i} \mid I_i]}$$

> Objective Function

From the budget constraint we have,

$$E[\tilde{W}_{1i} \mid I_i] = (1+r) W_{0i} + \{E[\tilde{P}_1 \mid I_i] - (1+r) P_0\} X_i$$

$$var[\tilde{W}_{1i} \mid I_i] = X_i^2 \, var[\tilde{P}_1 \mid I_i]$$

• Subbing in and computing the demand functions gives:

$$\boxed{X_i^d = \frac{E[\tilde{P}_1 \mid I_i] - (1+r)P_0}{a \cdot var[\tilde{P}_1 \mid I_i]}}$$

> Demand Functions

• Equating supply + demand gives the equilibrium condition:

$$\boxed{\sum_{i=1}^{n} \left\{ \frac{E[\tilde{P}_1 \mid y_i, P_0(y)] - (1+r)P_0(y)}{a \cdot var[\tilde{P}_1 \mid y_i, P_0(y)]} \right\} = \bar{X}}$$

> Market Clearing Condition

where $y = (y_1, y_2, \ldots, y_n)$. This equation determines the market-clearing price, $P_0(y)$. The main result is summarized by

Theorem: If $P_0(y)$ is given by

$$\boxed{P_0(y) = \beta_0 + \beta_1 \bar{y}}$$

> Equilibrium Price

where
$$\bar{y} = \frac{1}{n} \sum y_i$$

$$\beta_0 = \frac{\bar{P}_1 - a\sigma^2 \bar{X} \frac{1}{n}}{(1+r)(1+n\sigma^2)}$$

$$\beta_1 = \frac{n\sigma^2}{(1+r)(1+n\sigma^2)}$$

Then $P_0(y)$ is an equilibrium price. Hence, the equilibrium price reveals the average of all traders' signals. It aggregates the information in the economy in the sense that this would also be the price if each individual observed all the signals.

• To verify this conjectured equilibrium, we need 2 technical results:

Lemma 1: Let $h_i(y_i, \bar{y} \mid P_1)$ be the joint pdf of $\bar{y}$ and y_i, conditional on P_1. Then $\bar{y}$ is a sufficient statistic for $h_i(y_i, \bar{y} \mid P_1)$. That is, there exists functions $g_1(\cdot)$ & $g_2(\cdot)$ such that,

$$h_i(y_i, \bar{y} \mid P_1) = g_1(y_i, \bar{y}) \cdot g_2(\bar{y}, P_1)$$

Proof: Conditional on P_1 we know, $(y_i, \bar{y}) \sim N\left(\begin{pmatrix} P_1 \\ P_1 \end{pmatrix}, \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n \end{pmatrix}\right)$

Therefore,

$$h_i(y_i, \bar{y} \mid P_1) = \frac{1}{2\pi} \begin{vmatrix} 1 & 1/n \\ 1/n & 1/n \end{vmatrix}^{-1/2} \times \exp\left\{-\frac{1}{2} \begin{pmatrix} y_i - P_1 \\ \bar{y} - P_1 \end{pmatrix}' \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n \end{pmatrix}^{-1} \begin{pmatrix} y_i - P_1 \\ \bar{y} - P_1 \end{pmatrix}\right\}$$

$$= \frac{1}{2\pi} \frac{n}{\sqrt{n-1}} \exp\left\{-\frac{1}{2} \frac{n}{n-1} \Big[(y_i - P_1)^2 - (\bar{y} - P_1)(y_i - P_1) + (P_1 - y_i)(\bar{y} - P_1) + n(\bar{y} - P_1)^2 \right\}$$

If we let
$$g_1 = \frac{1}{2\pi} \frac{n}{\sqrt{n-1}} \exp\left\{-\frac{1}{2} \frac{n}{n-1} (y_i^2 - 2\bar{y} y_i)\right.$$
$$g_2 = \exp\left\{-\frac{1}{2} \frac{n}{n-1} \left[2 P_1 \bar{y} - P_1^2 + n(\bar{y} - P_1)^2\right]\right\}$$

Then the result follows.

This result can be used to prove the following,

Lemma 2: Let $m(P_1|\bar{y})$ be the density of P_1 conditional on $\bar{y}$, and let $\hat{m}(P_1|\bar{y}, y_i)$ be the density of P_1 conditional on $(\bar{y}, y_i)$. Then $m(P_1|\bar{y}) = \hat{m}(P_1|\bar{y}, y_i)$, and so therefore, $E(\tilde{P}_1|\bar{y}) = E[\tilde{P}_1|\bar{y}, y_i]$ and $var(\tilde{P}_1|\bar{y}) = var(\tilde{P}_1|\bar{y}, y_i)$.

- This result says that if traders know $\bar{y}$, their own private signal provides no additional info about $\tilde{P}_1$.

Proof: From Bayes Theorem,

$$\hat{m}(P_1|\bar{y}, y_i) = \frac{h_i(y_i, \bar{y}|P_1)\, g(P_1)}{\int_{-\infty}^{\infty} g(P_1) h_i(y_i, \bar{y}|P_1)\, dP_1}$$

where $g(P_1)$ is the marginal density (prior) of P_1. From Lemma 1,

$$\hat{m}(P_1|\bar{y}, y_i) = \frac{g(P_1) g_1(y_i, \bar{y}) g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1) g_1(y_i, \bar{y}) g_2(\bar{y}, P_1)\, dP_1} = \frac{g(P_1) g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1) g_2(\bar{y}, P_1)\, dP_1}$$

Let $f(\bar{y}|P_1)$ be the density of $\bar{y}$ given P_1. Again using Lemma 1,

$$f(\bar{y}|P_1) = \int_{-\infty}^{\infty} h_i(y_i, \bar{y}|P_1)\, dy_i = g_2(\bar{y}, P_1) \int_{-\infty}^{\infty} g_1(y_i, \bar{y})\, dy_i$$

Again from Bayes Theorem,

$$m(P_1|\bar{y}) = \frac{g(P_1) f(\bar{y}|P_1)}{\int_{-\infty}^{\infty} g(P_1) f(\bar{y}|P_1)\, dP_1} = \frac{g(P_1) g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1) g_2(\bar{y}, P_1)\, dP_1}$$

which is the same as $\hat{m}(P_1|\bar{y}, y_i)$!

• We can now finally prove the main result. From Lemma 2, all traders end up having the same expectations,

$$E[\tilde{P}_1 | y_i, P_0(y)] = \frac{\bar{P}_1 + n\sigma^2 \bar{y}}{1 + n\sigma^2}$$

$$\text{var}[\tilde{P}_1 | y_i, P_0(y)] = \frac{\sigma^2}{1 + n\sigma^2}$$

where we have used Facts 1 + 2. Subbing into the demand functions

$$\sum_{i=1}^{n} X_i^d[P_0, y_i] = \sum_{i=1}^{n} \left\{ \frac{\frac{\bar{P}_1 + n\sigma^2 \bar{y}}{1+n\sigma^2} - (1+r)(\beta_0 + \beta_1 \bar{y})}{a \frac{\sigma^2}{1+n\sigma^2}} \right\}$$

Subbing in for the conjectured β_0 and β_1 gives

$$\sum_{i=1}^{n} X_i^d[P_0, y_i] = \frac{\frac{\bar{P}_1 + n\sigma^2 \bar{y}}{1+n\sigma^2} - (1+r)\left[\frac{\bar{P}_1 - a\sigma^2 \bar{x} \frac{1}{n}}{(1+n\sigma^2)(1+r)} + \frac{n\sigma^2}{(1+r)(1+n\sigma^2)} \bar{y}\right]}{a \frac{\sigma^2}{1+n\sigma^2}}$$

$$= \bar{X}$$

We're done!

Comments

1.) It seems odd that demand doesn't depend on P_0. This is because these are equilibrium demands. Prices are playing a dual role here. Higher P_0 reduces demand for the usual (substitution effect) reasons. However, a higher P_0 also signals higher expected payoff. The two effects exactly cancel here.

2.) Even more strange is the fact that individual demands do not depend on individual signals, y_i. Isn't this a contradiction? If demands don't depend on y_i, then how does P_0 become a function of $\bar{y}$! [Grossman presents an out-of-equilibrium tatonnment argument].

3.) Clearly, if costs something to observe signals, then no equilibrium exists:

a.) If prices reveal all info, then nobody pays to acquire info.

b.) If nobody acquires info., then prices are uninformative, and if the signal cost is sufficiently low, an individual trader would want to acquire info.

This is the "paradox" that Grossman + Stiglitz address in their paper.

4.) The model has natural + intuitive comparative statics predictions:

$$\bar{x}\uparrow \implies P_0 \downarrow$$

$$a\uparrow \implies P_0 \downarrow$$

$$\lim_{n\to\infty} P_0 = \frac{P_1}{1+r} \quad (\bar{y} \to P_1 \text{, asset becomes riskless}).$$