# Topics for Today

1.) Grossman & Stiglitz (1980)

- Motivation
- Assumptions
- Equilibrium for fixed $\lambda$
- Equilibrium $\lambda$
- Comparative Statics
- Extensions

# Motivation

- Last time we studied how asset markets aggregate information. Today we study how asset markets transmit information from informed traders to uninformed traders.

- Last time we assumed traders observed signals "for free". Now we assume signals are costly.

- Clearly, if information is costly, then prices cannot transmit information perfectly, otherwise no one would buy it. We must look for an equilibrium where informed traders make just enough (expected, risk-adjusted) profits from trading with the uninformed traders to cover their information costs. By definition, in equilibrium everyone will be indifferent between becoming informed and remaining uninformed.

- The key result is that if information is costly to produce and acquire, then financial markets must be "noisy". GS show just how noisy they must be.

- The GS model has become a workhorse framework for studying many interesting questions in economics, finance, & accounting.

# Assumptions

1.) Static, 2-period model. Invest today, consume tomorrow.

2.) 2 assets: Safe asset with return $R$
Risky asset with return $u = \theta + \varepsilon$

3.) $\theta$ is observable with cost $c$. $\varepsilon$ is unobservable.

4.) 2 types of traders: Informed (observe $\theta$ and $P$)
Uninformed (only observe $P$).

5.) Traders are ex ante identical, with utility functions.

$$V(W_1) = -e^{-aW_1} \qquad a: \text{CARA}$$

6.) The (per capita) supply of the risky asset is random

$$S = \bar{x} + x$$

Traders know $\bar{x}$, but not $x$. [$x$ can be interpreted as arising from noise or liquidity traders]. Random $x$ is important since it prevents prices from revealing $\theta$ to uninformed. [It also breaks the "No Trade Theorem" (Tirole (1982)].

7.) The underlying randomness in the economy is $(\theta, \varepsilon, x)$. They are jointly normal + independent. Traders share common priors, $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, $x \sim N(0, \sigma_x^2)$.

# Equilibrium for Fixed $\lambda$

- The timing is as follows:
  1.) Traders decide whether to become informed
  2.) Conditional on info choice + prices, agents purchase assets
  3.) $x$ is realized, and price adjusts to clear the market.
  4.) $\varepsilon / u$ is realized. Traders consume their wealth.
- Note, 2 + 3 occur simultaneously. That is, price plays a dual role: (i) It adjusts to clear the market, (ii) It conveys information.
- Our job is to calculate equilibrium values of the price ($P$), and the fraction of informed traders ($\lambda$).
- As usual, we work backwards. We first calculate equil. price as a function of $\lambda$. We then compute $\lambda$ so that the expected utilities of being informed + uninformed are equal.
- An informed trader's demand is simple, since
  $E(u \mid \theta, P) = \theta$.

$$X_I(P, \theta) = \frac{\theta - RP}{a \cdot \sigma_\varepsilon^2}$$

> Informed Trader's Demand

- Uninformed trader's demand has the same basic form, but the conditional mean + variance is a bit trickier, since he only gets to observe $P(\theta, x)$. That is, he has a nontrivial signal extraction problem to solve:

$$X_u(P) = \frac{E[u \mid P(\theta,x)] - R \cdot P}{a \cdot var[u \mid P(\theta,x)]}$$

> U trader's demand

- To solve the signal extraction problem, we employ a "guess + verify" strategy. Let's assume that equilibrium price has the following form (linear is a good guess since everything is Gaussian).

$$P = A + B \cdot \theta + C \cdot x$$

> Conjectured form of equil. price

where (A, B, C) are undetermined coefficients.

- Since P is observed (and its functional form is known), the U traders effectively get the following noisy signal of $\theta$

$$\frac{P - A}{B} = \theta + \frac{C}{B} x$$

- We can now apply our Bayesian update rules from last time:

$$E[\theta \mid P] = \frac{\sigma_\theta^{-2}}{\sigma_{\theta|P}^{-2}} \bar{\theta} + \frac{(C/B)^{-2} \sigma_x^{-2}}{\sigma_{\theta|P}^{-2}} \left( \frac{P - A}{B} \right)$$

> Posterior Mean

$$\sigma_{\theta|P}^{-2} = \sigma_\theta^{-2} + \left( \frac{C}{B} \right)^{-2} \sigma_x^{-2}$$

> Posterior Precision

- The market-clearing condition is:

$$\boxed{\lambda \cdot X_I(P,\theta) + (1-\lambda)\cdot X_U(P) = \bar{X} + x}$$ > Equilibrium Condition

- Substituting in the above results,

$$\lambda \cdot \left[\frac{\theta - R(A + B\theta + Cx)}{a \cdot \sigma_\varepsilon^2}\right]$$

$$+ (1-\lambda)\cdot\left[\frac{\sigma_{\theta|P}^2\left(\sigma_\theta^{-2}\bar{\theta} + (B/C)^2\sigma_x^{-2}\right)\left(\theta + \frac{C}{B}x\right) - R(A + B\theta + Cx)}{a\cdot\left(\sigma_{\theta|P}^2 + \sigma_\varepsilon^2\right)}\right] = \bar{X} + x$$

where we've used the fact that $\mathrm{var}[u|P] = \sigma_{\theta|P}^2 + \sigma_\varepsilon^2$.

- Matching coefficients gives the following <u>Fixed Point Conditions</u>:

1.) $$\lambda \cdot \frac{(1-RB)}{a\cdot\sigma_\varepsilon^2} + (1-\lambda)\left[\frac{\sigma_{\theta|P}^2\left(\sigma_\theta^{-2}\bar{\theta} + (\frac{B}{C})^2\sigma_x^{-2}\right) - RB}{a\cdot\left(\sigma_{\theta|P}^2 + \sigma_\varepsilon^2\right)}\right] = 0$$ > coefficient on $\theta$

2.) $$\lambda \cdot \left[\frac{-RC}{a\,\sigma_\varepsilon^2}\right] + (1-\lambda)\left[\frac{\sigma_{\theta|P}^2\left(\sigma_\theta^{-2}\bar{\theta} + (\frac{B}{C})^2\sigma_x^{-2}\right) - RC}{a\cdot\left(\sigma_{\theta|P}^2 + \sigma_\varepsilon^2\right)}\right] = 1$$ > coefficient on $x$

3.) $$\lambda \cdot \left[\frac{-RA}{a\,\sigma_\varepsilon^2}\right] + (1-\lambda)\left[\frac{-RA}{a\cdot\left(\sigma_{\theta|P}^2 + \sigma_\varepsilon^2\right)}\right] = \bar{X}$$ > constant

# Comments

1.) Note that (3) easily gives $A$ as a function of $(B, C)$. Not surprisingly, it is a decreasing function of $\bar{x}$.

2.) Less obviously, note that if $C = \frac{-a\sigma_\varepsilon^2}{\lambda} B$, then (1) and (2) become identical. In fact, subbing out $C$, either can be used to solve for $B$. The solution is unique since the resulting equation is linear.

3.) Hence, the equilibrium price can be written

$$P = A + B\left(\theta - \frac{a\sigma_\varepsilon^2}{\lambda} x\right).$$

4.) Note that the noisy asset supply, $x$, prevents U traders from inferring $\theta$ from $P$.

5.) The amount of info contained in $P$ about $\theta$ depends on $\sigma_x^2$ and $\frac{a\sigma_\varepsilon^2}{\lambda}$. If $\sigma_x^2$ is big, then $P$ is very noisy + doesn't reveal much about $\theta$. More interestingly, when $a$ and $\sigma_\varepsilon^2$ are small, I traders react strongly to $\theta$, which conveys a lot of info. Similarly, when $\lambda$ is big, there are a lot of I traders, so $P$ becomes informative.

6.) One can verify than $B$ is increasing in $\lambda$, so that $P$ responds more to $\theta$ when there are a lot of informed traders. This will limit the number of I traders.

# Equilibrium $\lambda$

- We are only half done. We must now determine equilibrium in the "information market". That is, we must now calculate the equilibrium fraction of informed traders, $\lambda$.

- Before any uncertainty is resolved, all agents decide (simultaneously) whether to become informed. Equil. requires expected utilities to be the same for both.

- The exp. util. of U traders is relatively simple. Let $\hat{u} = E[u \mid P]$. Then,

$$W_u^1 = RW_u^0 + [u - RP] \cdot X_u \qquad X_u = \frac{\hat{u} - RP}{a \cdot (\sigma^2_{\theta|P} + \sigma^2_\varepsilon)}$$

- Conditional on P, $X_u$ is just a number, so we can apply our formula for the mean of a lognormal

$$E[U_u \mid P] = -e^{-aRW_u^0} \, e^{-\frac{(\hat{u} - RP)^2}{\sigma^2_{\theta|P} + \sigma^2_\varepsilon} + \frac{1}{2}\left(\frac{\hat{u} - RP}{\sigma^2_{\theta|P} + \sigma^2_\varepsilon}\right)^2 (\sigma^2_{\theta|P} + \sigma^2_\varepsilon)}$$

$$= -e^{-aRW_u^0} \cdot \exp\left\{-\frac{1}{2}\frac{(\hat{u} - RP)^2}{\sigma^2_{\theta|P} + \sigma^2_\varepsilon}\right\}$$

- For I traders we have

$$E[U_I \mid P] = -e^{-aRW_I^0} \, E\left[\exp\left\{-\frac{1}{2}\frac{(\theta - RP)^2}{\sigma^2_\varepsilon}\right\} \Big| P\right]$$

I traders must evaluate this, since they don't observe $\theta$ before paying the info cost

- To do this, we need the following fact:

**Fact**: If $x \sim N(\mu, \sigma^2)$ then $E e^{-tx^2} = \frac{1}{\sqrt{1+2\sigma^2 t}} e^{-\frac{\mu^2 t}{1+2\sigma^2 t}}$

This can be proved by a standard "complete-the-square" trick.

- Using this we have

$$E[U_I | P] = -e^{-aRW_I^0} \sqrt{\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_{\theta|P}^2}} \exp\left\{-\frac{1}{2}\frac{(\hat{\mu} - Rp)^2}{\sigma_\varepsilon^2 + \sigma_{\theta|P}^2}\right\}$$

- If $W_I^0 = W_u^0$ we can then write

$$E[U_I | P] - E[U_u | P] = \left(\sqrt{\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_{\theta|P}^2}} - 1\right) E[U_u | P]$$

- The final step is to remember that I traders must pay a cost c to observe $\theta$, so their net utility is

$$U_I = -e^{-a(W' - c)} = -e^{ac} e^{-aW'}$$

so that,

$$E[U_I | P] - E[U_u | P] = \left(e^{ac}\sqrt{\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_{\theta|P}^2}} - 1\right) E[U_u | P] = 0$$

Equilibrium Condition in Info Market

- This implies the following equilibrium condition,

$$\sqrt{\frac{\sigma_\varepsilon^2 + \sigma_{\theta|P}^2}{\sigma_\varepsilon^2}} = e^{ac}$$

- From our previous results we know,

$$\sigma_{\theta|P}^2 = \frac{1}{\sigma_\theta^{-2} + \sigma_x^{-2}\frac{\lambda^2}{(a\sigma_\varepsilon^2)^2}}$$

Hence, as $\lambda \uparrow \Rightarrow \sigma_{\theta|P}^2 \downarrow$. This implies info. choice is a strategic substitute. The more informed traders there are, the less valuable it is to be an informed trader

- Following GS, we can define the following

$$n = \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$$ } quality of info $\quad corr(\theta, u) = \sqrt{\frac{n}{1+n}}$

$$m = \frac{\sigma_x^2}{\sigma_\theta^2}\left(\frac{a\sigma_\varepsilon^2}{\lambda}\right)^2$$ } informativeness of the price system (inversely) $\quad \rho = corr(P, \theta) = \frac{1}{\sqrt{1+m}}$

- We can then write the equilibrium condition as

$$\boxed{\frac{m}{1+m} = 1 - \rho^2 = \frac{e^{2ac} - 1}{n}}$$

> Equil. in Info. Mkt.

# Comparative Statics

1.) An increase in the quality of information ($n \uparrow$) increases the informativeness of the price system ($\rho^2 \uparrow$).

2.) A decrease in the info cost ($c \downarrow$) increases the informativeness of the price system ($\rho^2 \uparrow$).

3.) Lower risk aversion ($a \downarrow$) increases the informativeness of the price system ($\rho^2 \uparrow$)

4.) As noise increases ($\sigma_x^2 \uparrow$), more traders become informed ($\lambda \uparrow$). In equilibrium, the informativeness of the price system remains the same.

5.) As $\sigma_x^2 \to 0$, $\lambda \to 0$. Equilibrium does not exist in the limit if $e^{ac} < \sqrt{1+n}$

6.) Trading volume between U and I goes to 0 as $\lambda \to \{0, 1\}$. Trading volume is maximized for intermediate values of $\lambda$. Volume depends on "heterogeneous beliefs".

Note $\lambda \cdot X_I + (1-\lambda) X_u = \bar{x} + x$

$$\Rightarrow \lambda \cdot (x_I - x) + (1-\lambda)(x_u - x) = \bar{x}$$

$$\boxed{E\, \lambda \cdot (x_I - x) = \frac{\lambda \cdot (1-\lambda)\, m\, \bar{x}}{1 + m + \lambda m n}}$$

> Average Trading Volume (net of noisy asset supply)

# Extensions

1.) Allow traders to choose precision of their signals, rather than an all-or-nothing choice. More precise signals are more expensive. [Verrecchia (1982)].

2.) Allow informed traders to have market power. (i.e, "insider trader"). Strategic Behavior [Kyle (1985)].

3.) Dynamic, Multiple time periods. [Wang (1993, 1994)].

4.) Multiple Signals / Assets. Which assets should I become informed about? [Van Nieuwerburgh + Veldkamp (2009)].