Topics for Today

1.) The Stochastic HJB Equation

2.) Examples
- The stochastic Linear-Quadratic Regulator (LQR)
- A Simple Growth Model

3.) Intro. to Arrow-Debreu General Equilibrium

- Let's use Ito's Lemma to derive the stochastic HJB eq.

- Now our value function is

$$V(x) = E_t \int_t^{\infty} e^{-r(s-t)} f(x,u)\, ds$$

- Following the same steps as before:

$$V(x) = \max_u \left\{ f(x,u)\Delta t + \frac{1}{1+r\cdot\Delta t} E\left[V(x+\Delta x) \mid x,u \right] \right\}$$

- Multiply by $1+r\cdot\Delta t$, subtract $V(x)$ from both sides, divide by Δt, and let $\Delta t \to 0$

$$rV(x) = \max_u \left\{ f(x,u) + \frac{1}{dt} E\left[dV \right] \right\}$$

- Suppose $dx = \mu(x,u)\cdot dt + \sigma(x)\, dw$

- From Ito's Lemma,

$$dV = V_x\, dx + \frac{1}{2}\sigma^2(x) V_{xx}\, dt$$

$$= V_x\left[\mu(x,u) + \sigma(x)\, dw \right] + \frac{1}{2}\sigma^2(x) V_{xx}\, dt$$

$$E[dV] = \left[V_x \cdot \mu(x,u) + \frac{1}{2}\sigma^2(x) V_{xx} \right] dt$$

> since $E[dw] = 0$

Therefore,

$$\boxed{rV = \max_u \left\{ f(x,u) + \mu(x,u)\cdot V_x + \frac{1}{2}\sigma^2(x) V_{xx} \right\}}$$

> Stochastic HJB Eq.

- Note, this is a 2nd-order ODE.

Example 1: The Stochastic LQR

- Let's return to our earlier example, but now suppose the state evolves randomly,

$$\min_u E \int_0^\infty e^{-rt}(ax^2 + bu^2)\,ds$$

$$\text{s.t. } dx = (cx + u)\,dt + \sigma\,dw$$

Stationary HJB Equation

$$\boxed{rV = \min_u \left\{ (ax^2 + bu^2) + (cx+u)\cdot V_x + \tfrac{1}{2}\sigma^2 V_{xx} \right\}}$$

FOC (u): $2bu + V_x = 0 \Rightarrow u = -\frac{1}{2b} V_x$

Sub u back into HJB

$$rV = ax^2 + \frac{1}{4b}V_x^2 + cx \cdot V_x - \frac{1}{2b}V_x^2 + \frac{1}{2}\sigma^2 V_{xx}$$

Guess : $V(x) = Ax^2 + B$ $\Rightarrow V_x = 2Ax \quad V_{xx} = 2A$

sub guess into HJB,

$$r(Ax^2+B) = ax^2 - \frac{1}{b}A^2x^2 + 2Acx^2 + \frac{1}{2}\sigma^2 A$$

Match Coefficients

1.) $rA = a - \frac{1}{b}A^2 + 2Ac$

$\Longrightarrow \frac{1}{b}A^2 + (r-2c)A - a = 0$ } Same as before!

2.) $rB = \sigma^2 A \Longrightarrow B = \dfrac{\sigma^2 A}{r}$

Comments

1.) Note that the optimal policy is the same as before, $u = -\frac{A}{2b}x$. Why? Why doesn't 'risk' matter? Does this mean risk is irrelevant?

2.) Verify that if instead,

$$dx = (cx+u)dt + \sigma x\, dw$$ } Note that x now affects the variance of the shocks.

then σ would influence behavior.

Example 2: A Simple Stochastic Growth Model

- Consider the following growth model featuring a linear/stochastic technology and CRRA preferences:

$$\max_c E \int_0^\infty e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt$$

$$\text{s.t. } dk = (\mu k - c) \cdot dt + \sigma k \, dw$$

Stationary HJB Eq.

$$\boxed{\rho V = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + (\mu k - c) \cdot V_k + \frac{1}{2}\sigma^2 k^2 \cdot V_{kk} \right\}}$$

FOC (c): $c^{-\gamma} = V_k \Rightarrow c = (V_k)^{-1/\gamma}$

$$c^{1-\gamma} = V_k^{1-1/\gamma}$$

Guess: $V(k) = \frac{A}{1-\gamma} k^{1-\gamma} \Rightarrow V_k = A k^{-\gamma}$

$$V_{kk} = -\gamma A k^{-\gamma-1}$$

Sub into HJB

$$\rho \frac{A}{1-\gamma} k^{1-\gamma} = \frac{1}{1-\gamma} A^{1-1/\gamma} k^{1-\gamma} + \mu A k^{1-\gamma} - A^{1-1/\gamma} k^{1-\gamma} - \frac{1}{2}\sigma^2 \gamma A k^{1-\gamma}$$

Cancel $Ak^{1-\gamma}$ from both sides

$$\rho = A^{-1/\gamma} + \mu(1-\gamma) - (1-\gamma)A^{-1/\gamma} - \tfrac{1}{2}\sigma^2\gamma(1-\gamma)$$

Note, $c = (V_k)^{-1/\gamma} = A^{-1/\gamma} k$

So let's solve for $A^{-1/\gamma}$

$$A^{-1/\gamma} = \frac{1}{\gamma}\left[\rho - (1-\gamma)\mu + \tfrac{1}{2}\sigma^2\gamma(1-\gamma)\right]$$

Therefore,

$$\boxed{\begin{aligned} c &= \left[\mu + \tfrac{1}{\gamma}(\rho - \mu) - \tfrac{1}{2}\sigma^2(\gamma-1)\right]k \quad \text{\} Policy Function} \\ dk &= \left[\tfrac{1}{\gamma}(\mu - \rho) + \tfrac{1}{2}\sigma^2(\gamma-1)\right]\cdot k\,dt + \sigma k\,dw \end{aligned}}$$

Comments

1.) The 2^{nd} term in the consumption function is an intertemporal substitution effect. The 3^{rd} term is a precautionary savings effect.

2.) If $\gamma = 1$ (log utility), then we get Friedman's Permanent Income Hypothesis, $c = \rho k$

3.) The usual condition for "endogenous growth" is that $\mu > \rho$. However, notice with uncertainty there can be sustained growth even if $\mu < \rho$, as long as $\gamma > 1$ and σ^2 is big enough.

Intro to Arrow-Debreu

- There are two big questions in economics:
  1.) How do markets + prices coordinate everyone's actions? How do we know they even can?
  2.) What are the normative implications of competive equilibria? Adam Smith told us that competition and free markets were good, but he never actually defined what "good" was?
- Arrow + Debreu were the first to formally prove that a general equilibrium set of prices even exists.
- Earlier attempts had just counted equations + unknowns!
- Their proof was based on a fixed point theorem.
- AD also clarified the normative implications of competitive equilibria - Under certain conditions, comp. equil. are Pareto Optimal.
- The AD model is an important theoretical benchmark. Most recent work in macro/finance attempts to remedy its limitations.
- The Key (Nobel prize-winning) idea in AD is to convert dynamic, stochastic settings into conventional (static) General Equilibrium theory by indexing goods by date and state (i.e., introduce additional goods called contingent claims).

Comments on the AD Model

1.) **Complete Markets**: The AD model presumes complete markets. If there are T periods, S states in each period, and N goods, then there must be $S \times N \times T$ markets.

2.) **Time**: The AD model is timeless. All ~~tra~~ trades take place at "time-0", before the resolution of uncertainty. (Rawls' "Veil of Ignorance"). After time-0, there are deliveries, but no trades. This really pushes the Walrasian auctioneer idea to the extreme!

3.) **Knowledge**: The relevant "state space" is known, and realizations of the state are verifiable. AD does not easily accommodate asymmetric info. or "Rumsfeld Uncertainty" (unknown unknowns).

4.) **Commitment**: No reneging/No Default. Ex post, after uncertainty is realized, an agent may not want to honor his AD contracts. It is assumed that AD contracts can be enforced. (Note: Since all trades occur at time-0, with known prices, and there are no unforeseen contingencies, unintentional default is not an issue).

5.) Dynamic Consistency: Since time-0 trading in a full set of AD state-contingent claims produces a Pareto Optimal allocation, if markets were to re-open in the future, no trades would occur! (Some person might want to trade, but if so, there would be no one else on the other side of the market).

6.) Market Power: All agents must be competitive. (i.e., price-takers).

7.) Money: There is no role for money in AD general equilibrium (other than as a unit-of-account).