Topics for Today

1.) The Expected Utility Hypothesis
  - Objective vs. Subjective Probabilities

2.) Risk Aversion

3.) Trade in Contingent Claims + the Allocation of Risk

4.) Asset Pricing with State-Contingent Claims
  - The "Law of One Price"

5.) The 1st + 2nd Welfare Theorems

6.) Arrow's (1964) Paper
  - "Arrow Securities

The Expected Utility Hypothesis

- The payoff on most assets is uncertain. When you buy an asset you are essentially buying a lottery. How do individuals value lotteries? This is obviously a crucial question for asset pricing!

- The St. Petersburg Paradox tells us that lotteries aren't valued according to expected payoff

St. Petersburg Paradox: Consider the lottery that pays 2^k with prob. $\frac{1}{2^k}$

prize	2	4	8	....
prob	1/2	1/4	1/8	

$\}$ How much would you pay for this lottery?

- We will (almost always) assume investors adhere to the Expected Utility Hypothesis.

$$\underbrace{U}_{\text{general utility function over lotteries}}(p \cdot x + (1-p) \cdot y) = \underbrace{p \cdot u(x) + (1-p) \cdot u(y)}_{\text{Expected Utility}}$$

$\}$ U is linear and additively separable.

- The Expected Utility Hypothesis is based on the following axioms:

1.) Continuity

2.) Completeness

3.) Independence

If $x \sim y$, then $p \cdot x + (1-p) \cdot z \sim p \cdot y + (1-p) \cdot z$

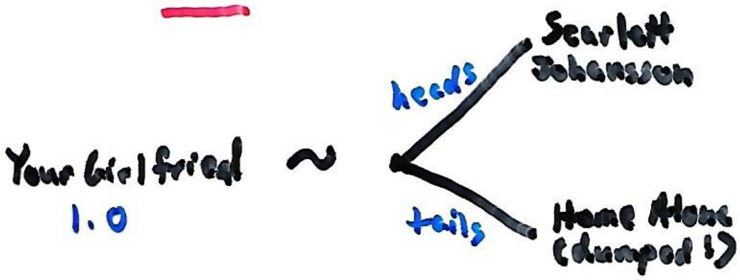
4.) Reduction of Compound Lotteries

- Sometimes a lottery payoff is just another lottery [e.g., derivative securities]. This says that all that matters is the final payoff probabilities.

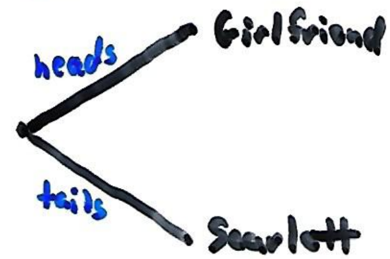
Caveats

- A lot of recent work explores relaxations of (3) and (4).
- The Independence Axiom seems reasonable, but consider the following pair of lotteries:

L1



L2



- Since gf(1.0) ~ [sj(heads); alone(tails)], if we replace gf in L2 and form **L3** 

, then according to the Independence Axiom (+ reduction of compound lotteries), **L2** ~ **L3**. Is it? Perhaps, if the coin flips are independent. But maybe not if there is correlation (consider the case of just one flip!)

- **Lesson**: A lottery considered in isolation may appear differently when packaged with other lotteries. This is the basis of both diversification and arbitrage.

- Many compound lotteries have a temporal aspect. The final payoff occurs after a sequence of previous lottery realizations. The Reduction of Compound Lotteries axiom implies that investors are indifferent to the timing of the resolution of uncertainty (controlling for time discounting). Experimental evidence suggests this isn't the case.

Objective vs. Subjective Probabilities: Von Neumann + Morgenstern assumed that lottery probabilities were objectively known (e.g., the spin of a roulette wheel). Often in econ + finance, the probabilities are unknown. (What's the prob. that the C\$ will reach par next year?) Savage (1954) showed how to extend the EUH to the case where probs. + utility functions are simultaneously constructed. The resulting probs. are subjective, and can differ across individuals. Savage needed to introduce a version of the Independence Axiom called the Sure Thing Principle.

Risk Aversion

- A key factor in asset pricing is risk aversion. According to the EUH, there are 2 (equivalent) ways to define and measure risk aversion:

1.) How much would you pay to avoid risk?

$$EU(\bar{x}+\varepsilon) = u(\bar{x} - \rho(\bar{x},\varepsilon))$$

Taylor Series: $u(\bar{x}+\varepsilon) \approx u(\bar{x}) + \varepsilon u'(\bar{x}) + \frac{1}{2}\varepsilon^2 u''(\bar{x})$

$$u(\bar{x}-\rho(\bar{x},\varepsilon)) \approx u(\bar{x}) - \rho(\bar{x},\varepsilon)u'(\bar{x})$$

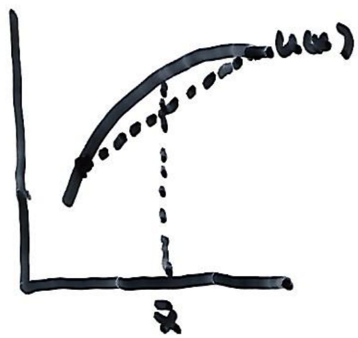
$$\Rightarrow \boxed{\rho(\bar{x},\varepsilon) = -\frac{1}{2}\sigma_\varepsilon^2 \frac{u''(\bar{x})}{u'(\bar{x})}}$$

$-\frac{u''}{u'}$ is called the Coefficient of Absolute Risk Aversion.

It measures the aversion to additive risks

Note, it depends on the level of wealth, $\bar{x}$.

2.) Curvature of the Von-Neumann/Morgenstern utility function, $u(x)$.



$\Rightarrow -\frac{u''}{u'}$ (Divide by u' in order to make unit free)

- Either way, a person is more risk averse when $u(x)$ is more concave (i.e., when $|u''|$ is bigger).

- A closely related concept is <u>relative</u> Risk Aversion. It measures aversion to <u>multiplicative</u> risks

$$E\,u(\bar{x}(1+\varepsilon)) = u(\bar{x}(1-\rho(\bar{x},\varepsilon)))$$

$$\Rightarrow \rho = -\tfrac{1}{2}\sigma_\varepsilon^2 \frac{\bar{x}\,u''(\bar{x})}{u'(\bar{x})}$$

- Again, relative risk aversion generally depends on the current level of wealth, $\bar{x}$.

- The following 2 utility functions are popular, since they lead to either constant absolute or constant relative risk aversion:

$$u(x) = -\frac{1}{\gamma}e^{-\gamma x}$$ } γ = Coef. of Absolute Risk Aversion
CONSTANT ABSOLUTE RISK AVERSION

$$u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$$ } γ = coef. of Relative Risk Aversion
CONSTANT RELATIVE RISK AVERSION

Calibrating Risk Aversion

- What share of your wealth are you willing to pay in order to avoid the risk of gaining or losing a share α of it with equal probability? [with CRRA, the answer doesn't depend on the level of wealth].

$$\frac{(1-\pi)^{1-\gamma}}{1-\gamma} = \frac{1}{2}\frac{(1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{(1+\alpha)^{1-\gamma}}{1-\gamma}$$

γ	$\alpha = .10$	$\alpha = .30$
1/2	0.3%	2.3%
1	0.5%	4.6%
4	2%	16%
10	4.4%	24.4%
40	8.4% !	28.7% !

- It's calculations like these that make many economists believe that a 'reasonable' value of γ should be less than about 10.

Contingent Claims Markets

- If markets are complete, AD state prices provide convenient vehicles for many asset pricing problems.
- For simplicity, assume just 2 states + 2 dates.

Problem

$$\max_{c_0,\, c(s)} \; u(c_0) + \beta \sum_s \pi(s) u(c(s))$$

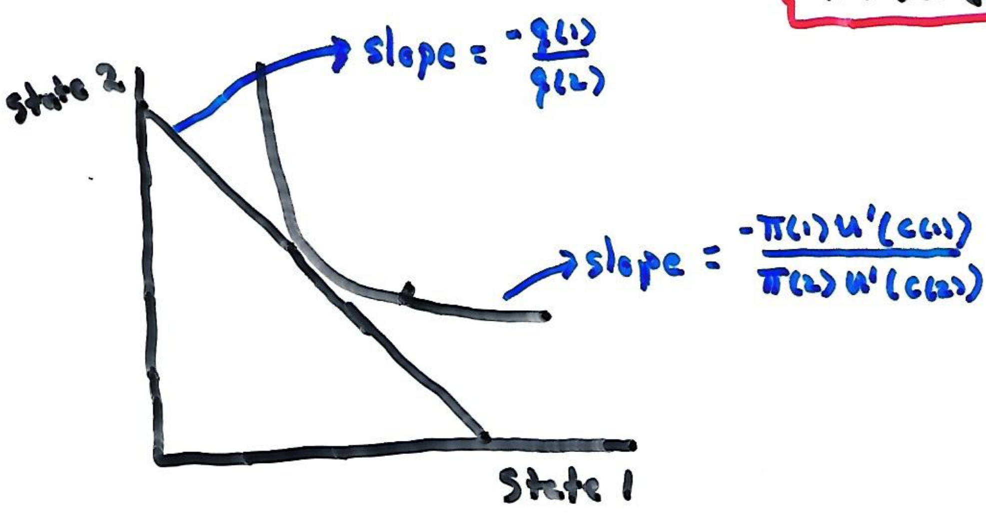
$$\text{s.t. } c_0 + \sum_s q(s) c(s) = y_0 + \sum q(s) y(s)$$

$y_0, y(s)$ are endowments

Equil. Conditions

1.) $u'(c_0) = \lambda$

2.) $\beta \pi(s) u'(c(s)) = \lambda q(s) \Rightarrow \boxed{\dfrac{\pi(1) u'(c(1))}{\pi(2) u'(c(2))} = \dfrac{q(1)}{q(2)}}$



Note, risk aversion is reflected in the curvature of the indifference curve.

- Note, $q(s)$ will be relatively high when:
  1.) $\pi(s)$ is high
  2.) $c(s)$ is low

Asset Pricing With State-Contingent Claims

$\boxed{\text{Asset} = \text{A bundle of state-contingent claims}}$ } Key Point!

"Law of One Price"

$$P(x) = \sum_s q(s) X(s)$$

asset price   payoff in state s

Spanning

- Let $\dim(s) = n$

  p = vector of n (observed) asset prices

  X = nxn payoff matrix, Rows are payoffs of each asset across states

  q = nx1 vector of (unobserved) AD state prices

From Law of One Price,

$$p = X \cdot q \Longrightarrow \boxed{q = X^{-1} \cdot p}$$

If X is full rank

- Therefore, with complete markets, traded assets can be used to "synthesize" underlying AD state prices.

The 1^{st} + 2^{nd} Welfare Theorems

• Let's briefly consider the normative implications of the AD model.

First Welfare Theorem: With complete markets, competitive equilibria are Pareto Optimal.

Comments

1.) Externalities are a form of incomplete markets. Hence, the theorem rules out (non-pecuniary) externalities.

2.) By definition, competitive equilibrium means all agents act as price-takers. Hence, the theorem rules out monopoly/monopsony power.

3.) Pareto Optimality is a very weak efficiency criterion. It ignores distributional issues + any notion of "distributive justice".

4.) This theorem is relatively easy to prove (by contradiction). For example, it is far easier than the proof of the existence of a competitive equilibrium.

Second Welfare Theorem: If prefences + technology are convex, any PO allocation can be supported (decentralized) as a Comp. Equil. with an appropriate initial redistribution of resources.

Comments

1.) This theorem can be used as a computational short-cut. Rather than solve a complex Fixed Point Problem (i.e., find prices that simultaneously clear all markets), we can compute an equil. in two steps: (1) Solve an optimization problem to find equil. quantities, and (2) Plug these quantities into the appropriate marginal conditions to find equil. prices.

2.) The 2nd welfare theorem is harder to prove. Convexity is required because the proof appeals to a Seperating Hyperplane Theorem.

Arrow's (1964) Paper

- Literally interpreted, the AD model is absurd. Time plays no role, and the required number of markets is beyond belief.

- Arrow shows that it is not as absurd as it might seem.

- Rather than trade once-and-for-all at some fictional time-0 in a huge number of markets, the same outcomes can be generated by a process of dynamic trading in a much smaller set of financial assets.

- Essentially, the ability to trade over time can take the place of explicit markets. This is the most important idea in all of asset pricing theory!

- The assets that Arrow invented are called (not surprisingly) "Arrow Securities". Modern financial engineers interpret observed assets as bundles of Arrow Securities.

- Although the paper contains a deep economic insight, in terms of the analysis, the paper mainly consists of notation and book-keeping. In fact, the main result (Theorem 2) ~~is~~ basically follows from the associative law of multiplication!

- Under a certain key assumption, the paper shows that you can decompose a single budget constraint consisting of $S \times C$ goods into a sum of two sequential budget constraints, one for S assets and another for C goods.

Notation

$\bar{P}_{sc}$ = time-0 price of commodity c in state s (to be consumed at time-1).

P_{sc} = time-1 price of commodity c if state s occurs

q_s = time-0 price of a \$ to be received in state s at time-1

- Note, $\bar{P}_{sc}$ will generally differ from P_{sc}. (Umbrellas are more expensive during a rainstorm).

Time-0 AD Budget Constraint

$$\sum^{S}\sum^{C} \bar{P}_{sc} x_{sc} = y = \sum^{S}\sum^{C} \bar{P}_{sc} w_{sc}$$ } S×C choices all at once

Sequential AD Budget Constraints

1.) Purchase $\sum^{C} P_{sc} x_{sc}$ claims to a \$ for **each** state $s = 1, 2, 3, \ldots$

Total Cost = $\sum^{S} q_s \cdot \sum^{C} P_{sc} x_{sc}$

2.) **After** the state is realized use asset payoffs to purchase (or supply) the C goods.

S choices at time-0 + C choices at time-1

- Note, the two problems are equivalent if

$$\boxed{q_s P_{sc} = \bar{P}_{sc}}$$

- In words,

$$\boxed{\text{Time-0 price of commodity } c \text{ in state } S = \begin{pmatrix}\text{Time-0 price of}\\ \$ \text{ in state } S\end{pmatrix} \times \begin{pmatrix}\text{Time-1 price of}\\ \text{commodity } c \text{ in}\\ \text{state } S\end{pmatrix}}$$

q_s : Reflects the ex ante likelihood of state S

P_{sc} : Reflects the ex post relative scarcity of goods in each state

- Hence, $\bar{P}_{sc}$ could be high either because q_s is high (e.g., rain is very likely), or because P_{sc} will be high (if it rains, there will be an umbrella shortage).

- The basic idea follows from the associative law of multiplication. For example, with 2 goods & 2 states:

$$q_1[P_{11}X_{11} + P_{12}X_{12}] + q_2[P_{21}X_{21} + P_{22}X_{22}]$$

$$= q_1 P_{11} X_{11} + q_1 P_{12} X_{12} + q_2 P_{21} X_{21} + q_2 P_{22} X_{22}$$

- At this point you might be asking yourself (following Keenan Thompson) - "What up wit dat?" Did Arrow just pull a fast one on us? Is it really that simple? In fact, it isn't. Arrow slipped in a strong and crucial assumption that we'll talk more about next time.