Topics for Today

1.) A Caveat to Arrow's Paper

2.) Dynamic Trading of Arrow Securities
   - Radner Equilibria

3.) Intro. to the CAPM
   - Portfolio Theory

A Caveat to Arrow

- Arrow shows that by allowing agents to trade over time, the number of markets required to support a complete markets allocation of risk is greatly reduced (from $C \times S$ to $C+S$).

- The key idea is that agents trade financial assets (Arrow Securities) before the resolution of uncertainty, and then trade goods & services after the resolution of uncertainty, using the proceeds from their asset portfolios.

- This seems to make the AD model more 'realistic'. However, Arrow (quietly) introduces a crucial assumption, which may not be so realistic.

- This additional assumption is required because when deciding on their portfolios at time 0, agents must know how much they will need to spend in the future. This will depend on future (spot) commodity prices. How are they supposed to know this?! Arrow simply assumes they do.

- Clarification: Sometimes people describe this as perfect foresight, which isn't right. Agents remain uncertain about future prices because they remain uncertain about the future state. In fact, agents can have different price expectations because they have heterogeneous prior beliefs about the future state.

- Instead, what they must know and agree on is the function mapping future states into future prices. Without this, there would be no guarantee that their ex ante plans would be consistent ex post.

Dynamic Trading of Arrow Securities

- Arrow confines his analysis to a 2-period world. Radner (1972) generalizes it to an arbitrary number of periods. A few new issues arise with ongoing trading:

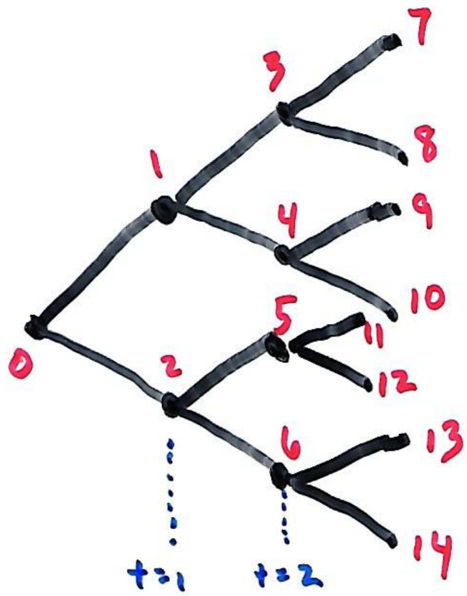
1.) How far ahead must agents look/forecast? Using the logic of dynamic programming, Radner shows you only need to look one period ahead.

2.) How many financial assets do you need? This depends on how information gets revealed over time. In some cases, a very small number of 'long-lived securities' will do the job.

3.) If agents have infinite horizons (or uncertain lifetimes) what's to prevent them from running a 'Ponzi Scheme'? (i.e., borrow new money to service existing debts). With a sequence of budget constraints, you must impose borrowing constraints.

4.) What happens to beliefs over time? Can heterogeneous beliefs be sustained, or do they eventually merge? This leads to the notion of a Rational Expectations Equilibrium.

• Radner assumes the world evolves according to an 'event tree'. For simplicity, consider the following binomial tree:



Standard AD ⟹ Need 15 (time-0) markets

With Arrow securities, need only 2!

S_u = pays 1 iff next period's state is up

S_d = pays 1 iff next period's state is down

Fact 1: Arrow securities can be interpreted as an 'orthogonal basis' for the 1-step ahead state space. (Any other linearly independent set of securities will work).

Fact 2: The Wiener Process is the continuous-time limit of a binomial tree.

Implication: If the underlying uncertainty in the economy evolves as a Brownian Motion, continuous trading of just two (independent) securities (e.g., Stock + Bond) spans the entire state space! All other securities are "redundant", and can be priced by no arbitrage.

• This is one reason why continuous-time models are popular in finance.

• **Borrowing Constraints**: An agent's holdings of Arrow securities will evolve over time in response to the resolution of uncertainty. His portfolio of Arrow securities can be interpreted as a form of '*wealth*', which becomes an endogenous state variable. If an agent's wealth is negative, he owes 'money', and he must never owe so much that he cannot pay it back even if he sets consumption to zero from then on!

• **Rational Expectations Equilibrium**: Even if agents' subjective beliefs initially differ, if their priors are '*mutually absolutely continuous*' (i.e., agree on 0 prob. events), then eventually their beliefs will *merge*, and their expectations will become the same. When they do, the outcome is called a "Rational Expectations Equilibrium". (Muth & Lucas slipped in one additional assumption - that 'the truth' is in the support of the prior, so that eventually everyone learns the correct model).

Portfolio Theory

- The AD model is very abstract. Its strength is its generality, and its focus on the fundamental role of financial markets in the economy.

- However, its generality means that it is difficult to generate many specific predictions from it.

- Coincidentally, around the same time Arrow was writing his paper, other (more applied) researchers were beginning to plant the seeds of a very practical approach to asset pricing.

- The starting point was Markowitz's (1952) work on portfolio theory. This focused on the problem of a single investor, taking asset returns as given. The next step was taken by Sharpe (1964), who embedded Markowitz's portfolio problem into a (static) equilibrium setting.

- We'll follow the same sequence, by first considering the (static) portfolio problem of a single investor.

- The key idea in all this is diversification. By holding a diversified portfolio you can "have your cake and eat it too" (i.e., get higher returns with less risk).

- Let's start by considering the simplest possible example of diversification - Suppose there are n assets with identical expected returns, $r_i = r$, $i = 1, 2, \dots n$, and identical variances, $\sigma_i^2 = \sigma^2$, $i = 1, 2, \dots n$. Moreover, suppose the returns are *uncorrelated*, and that the investor divides his portfolio equally among all assets:

$$\text{Expected Portfolio Return} = \frac{1}{n}\sum_{i=1}^{n} r_i = r$$

$$\text{Expected Portfolio Variance} = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 = \frac{\sigma^2}{n}$$

- Notice that by investing in each asset (i.e., by diversifying) you get the *same* expected return, but with *lower* risk (defined by the variance of the portfolio). Moreover, note that portfolio variance declines *rapidly* with the number of assets (with just 2 assets, it's cut in half, with 3 the variance is 1/3 as high, and so on).

- This example is the basis of the entire insurance industry!

- Rather than consider 1 investor and many assets, the same logic applies to the case of many investors and 1 asset:

True or False: If Bill & Ted have the same income + preferences, and there is a risky project which neither of them is willing to do on his own, they will not be willing to do it on a 50-50 basis either, since splitting the project halves both the return and riskiness of the project.

- Of course, in general, asset returns are correlated, and this has an important effect on the potential gains from diversification.

- Consider just 2 assets:

(r_1, σ_1^2) = expected return + variance of asset 1 (a1)

(r_2, σ_2^2) = expected return + variance of asset 2 (a2)

ρ : correlation between the returns of a1 and a2

α : share of portfolio invested in a2

Then,

$$r_p = (1-\alpha) r_1 + \alpha r_2$$ } Portfolio Return

$$\sigma_p = \sqrt{(1-\alpha)^2 \sigma_1^2 + 2\rho\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$ } Portfolio Standard Deviation