Topics for Today

1.) Example - The Gains from International Diversification

2.) A Diversification Fallacy
   - "Time Diversification"

3.) The General Markowitz Portfolio Problem
   - The "Mean-Variance Frontier"

4.) The Two-Fund Theorem

5.) Adding a Riskless Asset
   - The One-Fund Theorem

6.) Sharpe Ratios + the Price of Risk

7.) Mean-Variance Preferences + Optimal Portfolio Selection

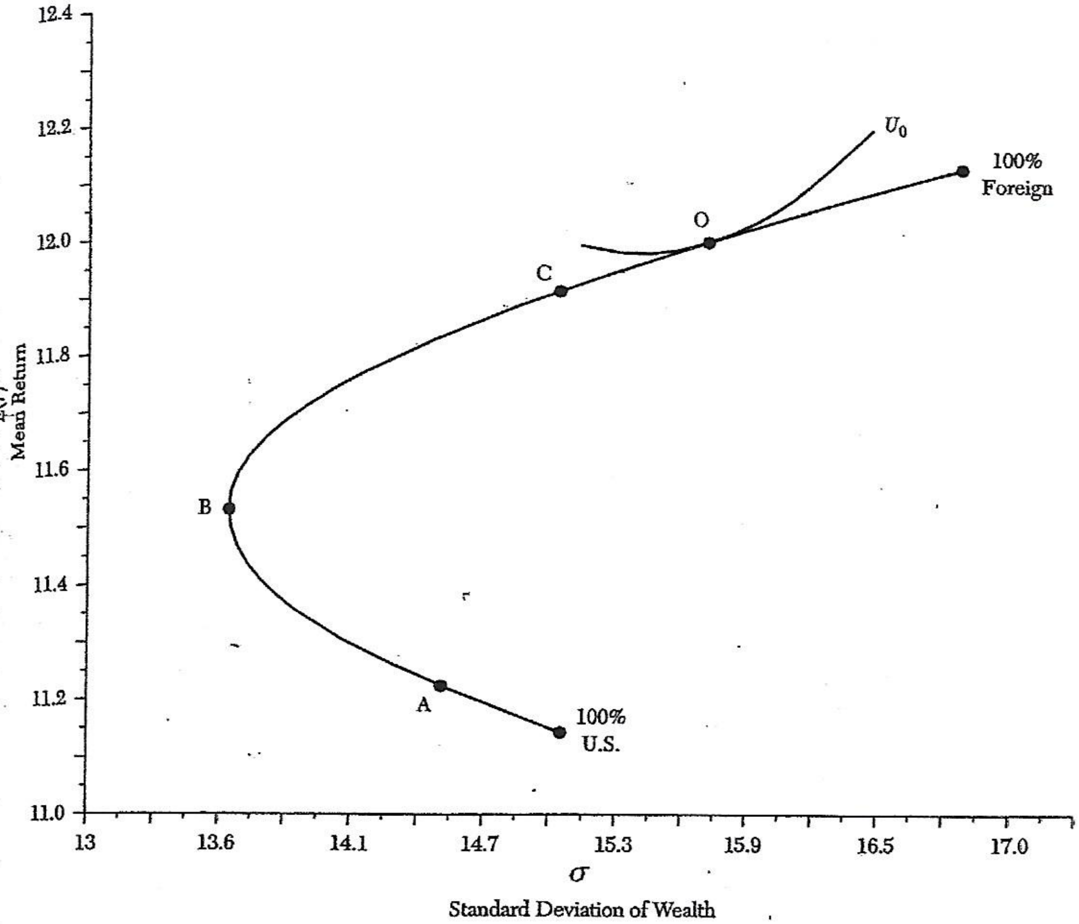


Figure 1. Risk Return Trade-Off Portfolios of U.S. and Foreign Mutual Funds

A Diversification Fallacy

- A common argument by investment professionals is the following:

  "Holding stocks over long time periods reduces risk, since you are diversifying over time".

- The intuition is supposedly that over long periods good and bad luck will "even out".

- However, given the assumptions of the CAPM, this argument is a fallacy. If returns are i.i.d. over time, a long holding period just means you are making the same bet over & over again. This adds risks, it doesn't average them. For example, the variance of a 10-year holding period is 10 times the variance of a 1-year holding period.

- The only way to make sense of the above advice is if there is mean reversion over time in asset returns. (That is, if returns were high this year, they are more likely to be lower in the future). In this case, long holding-periods do indeed lower variance.

- In fact, there is evidence of mean reversion in many asset returns. Unfortunately, mean reversion violates the assumptions of the (static) CAPM!

The General Markowitz Portfolio Problem

- Last time we saw the effects of diversification with just two assets. Now we generalize to n assets.

- Let r_i : expected return on asset i, $i = 1, 2, \cdots n$

  σ_{ij} = covariance between assets i and j, $i, j = 1, 2, \cdots n$

  w_i = portfolio weight of asset i

  r_p = expected portfolio return

  σ_p^2 = variance of portfolio return

- We want to minimize portfolio variance for a given expected portfolio return. The solution will then trace out the Mean-Variance Frontier as we vary r_p.

- Form the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda_1 \left[1 - \sum_{i=1}^{n} w_i \right] + \lambda_2 \left[r_p - \sum_{i=1}^{n} w_i r_i \right]$$

First-Order Conditions

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2 r_i = 0 \qquad i = 1, 2, \cdots n$$

$$\sum_{i=1}^{n} w_i = 1$$

$$\sum_{i=1}^{n} w_i r_i = r_p$$

- Note, these are $n+2$ linear equations in the $n+2$ unknowns $(w_i, \lambda_1, \lambda_2)$.

• Given the linearity, we can simplify using vector-matrix notation,

• Let w = $n \times 1$ vector of portfolio weights

Ω = $n \times 2$ variance-coveriance matrix of returns

$\bar{r}$ = $n \times 1$ vector of expected returns

$\mathbb{1}$ = $n \times 1$ vector of ones

$$\mathcal{L} = \tfrac{1}{2} w'\Omega w + \lambda_1 [1 - w'\mathbb{1}] + \lambda_2 [r_p - w'\bar{r}]$$

FOC(w): $\Omega w = \lambda_1 \mathbb{1} + \lambda_2 \bar{r}$

$$\Rightarrow w^* = \Omega^{-1}[\lambda_1 \mathbb{1} + \lambda_2 \bar{r}]$$

• sub into constraints,

$$1 = \mathbb{1}'w = \mathbb{1}'\Omega^{-1}[\lambda_1 \mathbb{1} + \lambda_2 \bar{r}] = \lambda_1 \mathbb{1}'\Omega^{-1}\mathbb{1} + \lambda_2 \mathbb{1}'\Omega^{-1}\bar{r}$$

$$r_p = \bar{r}'w = \bar{r}'\Omega^{-1}[\lambda_1 \mathbb{1} + \lambda_2 \bar{r}] = \lambda_1 \bar{r}'\Omega^{-1}\mathbb{1} + \lambda_2 \bar{r}'\Omega^{-1}\bar{r}$$

• Define the following 3 scalars,

$$a = \mathbb{1}'\Omega^{-1}\mathbb{1} \qquad b = \mathbb{1}'\Omega^{-1}\bar{r} \qquad c = \bar{r}'\Omega^{-1}\bar{r}$$

Then,

$$\begin{aligned} 1 &= \lambda_1 a + \lambda_2 b \\ r_p &= \lambda_1 b + \lambda_2 c \end{aligned} \Rightarrow \lambda_1 = \frac{c - b r_p}{\Delta} \quad \lambda_2 = \frac{a r_p - b}{\Delta}$$

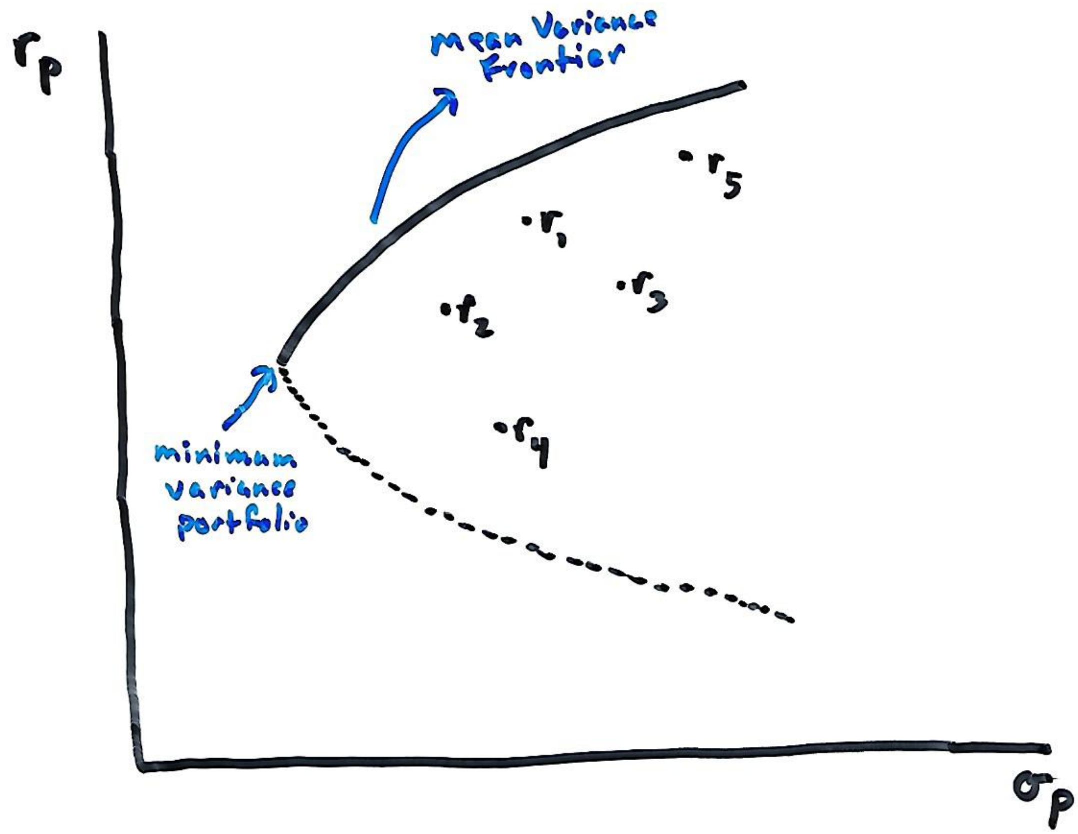
where $\Delta = ac - b^2$

• Therefore,

$$\boxed{\sigma_p^2 = w'\Omega w = \lambda_1 + \lambda_2 r_p = \frac{a r_p^2 - 2b r_p + c}{\Delta}}$$

• This is a parabola in (r_p, σ_p^2) space, or a hyperbola in (r_p, σ_p) space.

The Mean-Variance Frontier



Comments

1.) The MV Frontier can be thought of in two (equivalent) ways:

a.) Maximize expected return for given risk

b.) Minimize risk (variance) for given expected return

2.) The dotted portion of the minimum variance frontier is not efficient. An investor who values higher expected returns would always choose a portfolio on the upper portion of the min variance frontier (i.e., on the mean-variance frontier).

3.) Individual assets generally lie inside the frontier. Hence, by diversifying, you can achieve both higher returns and lower risk!

The Two-Fund Theorem

- The linearity of the equations characterizing the MV frontier imply the following surprising result:

Two-Fund Theorem: Any portfolio on the MV frontier can be achieved by taking linear combinations of just two portfolios on the frontier.

"Proof": This just follows from the fact that linear combos of solutions to linear systems of equations are also solutions. For example, consider the system $Ax = b$, and let x_1 and x_2 both be solutions. Define $x_3 = \alpha x_1 + (1-\alpha) x_2$. Then x_3 is also a solution, since $A(\alpha x_1 + (1-\alpha) x_2) = \alpha A x_1 + (1-\alpha) A x_2 = \alpha b + (1-\alpha) b = b$.

- This result is an example of a much more general result called the Principle of Superposition, which plays a big role in the theory of differential equations.

- **Practical Implication**: Efficient investment strategies only require two 'mutual funds'.

- This result leads to a simple algorithm for generating the MV frontier. Consider the 2 funds defined by $(\lambda_1 = 1, \lambda_2 = 0)$ and $(\lambda_1 = 0, \lambda_2 = 1)$. These produce the following 2 equations for the (unnormalized) weights:

$$v_1 = \Omega^{-1} \mathbb{1} \qquad v_2 = \Omega^{-1} \bar{r}$$

Normalize these so they sum to one

$$\tilde{\omega}_1 = \frac{\Omega^{-1} \mathbb{1}}{\mathbb{1}' \Omega^{-1} \mathbb{1}} \qquad \tilde{\omega}_2 = \frac{\Omega^{-1} \bar{r}}{\mathbb{1}' \Omega^{-1} \bar{r}}$$

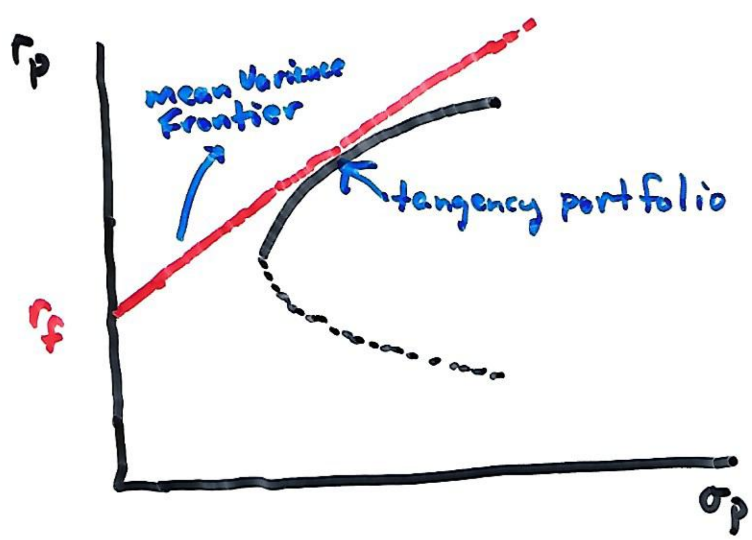
Then linear combos. of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ generate the entire MV frontier!

Adding a Riskless Asset

- In practice, investors often have access to a riskless investment (e.g., inflation-indexed T-bills). For simplicity, we assume that investors can lend and borrow at the riskless rate, r_f.

- Adding a riskless asset actually simplifies things a lot. It is visually obvious from the previous graph of the MV frontier that if we add a new asset with $(r_i, \sigma_i^2) = (r_f, 0)$, then the efficient frontier becomes a straight line

MV Frontier with Riskless Asset



- The tangency portfolio maximizes the ratio of portfolio excess return to portfolio risk. This maximum ratio is called the Price of Risk. More generally, a Sharpe Ratio is the ratio of excess returns to st. dev. for any given portfolio. Hence, the tangency portfolio delivers the maximum Sharpe ratio among all portfolios.

- To find the tangency portfolio we just need to modify the Lagrangian as follows, using the fact that now $r_p = w'\bar{r} + (1 - w'\mathbf{1}) r_f$

$$\mathcal{L} = \frac{1}{2} w' \Omega w + \lambda \left[r_p - w'\bar{r} - (1 - w'\mathbf{1}) r_f \right]$$

FOC(w): $\Omega w - \lambda(\bar{r} - r_f \cdot \mathbf{1}) = 0$

$$\Rightarrow \quad w = \Omega^{-1} \lambda (\bar{r} - r_f \cdot \mathbf{1})$$

Sub this into the constraint,

$$r_p - r_f = (\bar{r} - r_f \cdot \mathbf{1})' \lambda \Omega^{-1} (\bar{r} - r_f \cdot \mathbf{1})$$

$$\Rightarrow \quad \lambda = \frac{r_p - r_f}{(\bar{r} - r_f \cdot \mathbf{1})' \Omega^{-1} (\bar{r} - r_f \cdot \mathbf{1})}$$

Therefore,

$$w = \frac{r_p - r_f}{(\bar{r} - r_f \cdot \mathbf{1})' \Omega^{-1} (\bar{r} - r_f \cdot \mathbf{1})} \Omega^{-1} (\bar{r} - r_f \cdot \mathbf{1})$$

$$= \phi \bar{w}$$

$\Rightarrow$ All minimum variance portfolios are a multiple of a single portfolio of risky assets, $\bar{w}$.

- Normalizing $\bar{w}$ to sum to one

$$w_t = \frac{\Omega^{-1}(\bar{r} - r_f \cdot \mathbf{1})}{\mathbf{1}' \Omega^{-1} (\bar{r} - r_f \cdot \mathbf{1})}$$ } tangency portfolio

- Note that now we obtain a One-Fund Theorem. Efficient investment only requires shares in a single mutual fund of risky assets.

Mean-Variance Preferences + Optimal Portfolio Selection

- The optimal mix between the riskless asset and the tangency portfolio depends on preferences. The model is static, so that all that matters is end-of-period wealth. We assume investors only care about the mean + variance of end-of-period wealth

- There are 2 ways to motivate this assumption:

  1.) Quadratic Preferences, $U = r_p - \frac{1}{2}\gamma\sigma_p^2$

  2.) Returns are normally distributed, and preferences are exponential (CARA).

- Both have problems
  - Quadratic Preferences imply increasing absolute risk aversion
  - Empirically, returns are not normally distributed

- Let's proceed anyway. For simplicity, assume a single risky asset, (r_1, σ_1^2)

$$\max_{w_1} \; r_f + w(r_1 - r_f) - \frac{1}{2}\gamma w_1^2 \sigma_1^2$$

<u>FOC</u>: $\boxed{w_1 = \frac{r_1 - r_f}{\gamma\sigma_1^2}}$ } optimal investment in risky asset