Topics for Today

1.) The Merton (1971) Consumption/Portfolio Model

- Overview
- The Budget Constraint
- HARA Preferences
- Infinite Horizon Problem
- Finite Horizon Problem

2.) Extensions

- Labor Income
- Learning

Overview

- The CAPM is static. A natural question is whether it can be generalized to a dynamic + stochastic environment. This is the question Merton (1971) addresses.

- In a dynamic model there are two questions of interest:
  1.) How much should you save?
  2.) How should your savings be allocated among assets? (i.e., what should be your portfolio?).

- Like Sharpe, Merton mainly focuses on the case where all wealth is traded.

- Unlike Sharpe, Merton focuses on the Partial Equilibrium problem of a single investor. Asset returns are exogenous.

- The paper contains 3 Key results:
  1.) If asset prices evolve as a geometric Brownian Motion (i.e., returns are i.i.d. log-normal), then the 'Separation Theorems' of the CAPM apply. Investors can span the MV frontier with just 2 'mutual funds' (or just 1 fund of risky assets if there is a risk-free asset).

  2.) If preferences belong to the HARA class (Hyperbolic Absolute Risk Aversion), then decision rules are linear in wealth, and the model can be solved analytically.

  3.) If prices are Geometric Brownian Motion, and preferences have constant relative risk aversion, then the optimal portfolio is constant, even if agents have finite horizons (i.e., the portfolio policy is "myopic".)

The Budget Constraint

- As usual, we start with discrete time, then take limits. Assume 'periods' are of length h.
- Let $W(t)$ = wealth at the beginning of period t
  $N_i(t)$ = # of shares of asset i purchased during period t
  $c(t)$ = consumption rate during period t
- Then $W(t) = \sum_{i=1}^{n} N_i(t-h)P_i(t)$ } wealth = current market value of existing asset holdings

Budget Constraint

$$\sum^{n} N_i(t-h)P_i(t) = \sum^{n} N_i(t)P_i(t) + c(t)\cdot h$$ } $N(t)$, $c(t)$ chosen simultaneously

$$\Rightarrow -c(t)\cdot h = \sum^{n}[N_i(t) - N_i(t-h)]P_i(t)$$

$$\Rightarrow -c(t+h)\cdot h = \sum^{n}[N_i(t+h) - N_i(t)][P_i(t+h) - P_i(t)] + \sum^{n}[N_i(t+h) - N_i(t)]P_i(t)$$

Take limits ; $h \to 0$

$$-c(t)\cdot dt = \sum^{n} dN_i \cdot dP_i(t) + \sum^{n} dN_i(t)P_i(t)$$

Also, $W(t) = \sum^{n} N_i(t)P_i(t)$

From Ito, $dW = \sum^{n} N_i dP_i + \sum^{n} dN_i \cdot P_i + \sum dN_i \cdot dP_i$

$$\Rightarrow \boxed{dW = \sum N_i \cdot dP_i - c(t)dt}$$

- It is usually more convenient to work in terms of portfolio shares, rather than number of shares:

- Define $\omega_i = \frac{N_i(t) P_i(t)}{W(t)}$: Share of wealth allocated to asset i

- We can then write the budget constraint as follows:

$$dW = \sum^{n} \left(\frac{dP_i}{P_i}\right) \omega_i W - c(t)\,dt$$

- Suppose $\frac{dP_i}{P_i} = \alpha_i\,dt + \sigma_i\,dz_i$

- Then,

$$dW = \left[\sum^{n} \alpha_i W - c(t)\right]dt + \sum^{n} \omega_i \sigma_i\,dz_i$$

Special Case

Suppose there are just 2 assets

Asset 1 : $\frac{dP_1}{P_1} = \alpha \cdot dt + \sigma\,dz$ > risky

Asset 2 : $\frac{dP_2}{P_2} = r \cdot dt$ > riskless

$$\boxed{\begin{aligned} dW &= [\omega_1 \alpha + (1-\omega_1) r] W \cdot dt + \omega_1 \sigma\,dz \\ &= [r + \omega_1(\alpha - r)] W \cdot dt + \omega_1 \sigma\,dz \end{aligned}}$$

} "usual" budget constraint for Merton Problem

HARA Utility Functions

- Normally, the Merton problem does not possess an analytical solution
- However, when asset prices are Geometric Brownian Motion, and preferences are HARA, then you can obtain explicit solutions.

$$\boxed{\text{HARA}: \; u(c) = \alpha \frac{\gamma}{1-\gamma}\left(\beta + \frac{c}{\gamma}\right)^{1-\gamma}}$$

- The defining feature of HARA is linear risk tolerance.

$$\frac{-u'}{u''} = \beta + \frac{c}{\gamma}$$

The above utility function is just the solution of this diff. eq.

- HARA nests several popular preference specifications:

1.) $\lim_{\gamma \to \infty} \Longrightarrow \quad u(c) = -\beta e^{-c/\beta}$ } CARA

2.) $\beta = 0 \Longrightarrow \quad u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ } CRRA

3.) $\gamma = -1 \Longrightarrow \quad u(c) = -\frac{1}{2}(\beta - c)^2$ } Quadratic ($\beta > c$)

Infinite Horizon Problem

- Assume a riskless asset with rate of return, r, and a single risky asset which follows:

$$\frac{dP}{P} = \mu\, dt + \sigma\, dZ$$

- The investor's objective is:

$$\boxed{\begin{aligned} &\max_{C,\alpha} E\int_0^\infty e^{-\rho t}\frac{C^{1-\gamma}}{1-\gamma}\,dt \\ &\text{s.t. } dW = \left[(r+\alpha(\mu-r))W - C\right]dt + \alpha\sigma W\, dZ \end{aligned}}$$

α: share invested in risky asset

Stationary HJB

$$\rho V = \max_{C,\alpha}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \left[(r+\alpha(\mu-r))W - C\right]V_W + \frac{1}{2}\alpha^2\sigma^2W^2V_{WW}\right\}$$

FOCs

C: $\quad C^{-\gamma} = V_W \Rightarrow C = (V_W)^{-1/\gamma} \Rightarrow C^{1-\gamma} = (V_W)^{1-1/\gamma}$

α: $\quad (\mu-r)W\cdot V_W + \alpha\sigma^2W^2V_{WW} = 0$

$$\Rightarrow \alpha = \left(\frac{-V_W}{W\cdot V_{WW}}\right)\cdot\left(\frac{\mu-r}{\sigma^2}\right)$$

Guess : $V = AW^{1-\gamma}$

$$\Rightarrow V_W = (1-\gamma)AW^{-\gamma} \Rightarrow c = [(1-\gamma)A]^{-1/\gamma} W$$

$$V_{WW} = -\gamma(1-\gamma)AW^{-\gamma-1} \Rightarrow -\frac{V_W}{W \cdot V_{WW}} = \frac{1}{\gamma}$$

Sub back into HJB

$$\rho A W^{1-\gamma} = \frac{1}{1-\gamma}\left[(1-\gamma)A\right]^{1-1/\gamma} W^{1-\gamma} + \left[\left(r + \frac{(\mu-r)^2}{\gamma\sigma^2} - \left[(1-\gamma)A\right]^{-\frac{1}{\gamma}}\right](1-\gamma)AW^{1-\gamma}$$

$$- \frac{1}{2}\frac{(\mu-r)^2}{\sigma^4\gamma^2}\sigma^2\left[A\gamma(1-\gamma)\right]W^{1-\gamma}$$

Divide both sides by $AW^{1-\gamma}$

$$\rho = \frac{1}{1-\gamma}(1-\gamma)^{1-1/\gamma}A^{-1/\gamma} + (1-\gamma)\left[r + \frac{(\mu-r)^2}{\gamma\sigma^2}\right] - (1-\gamma)\left[(1-\gamma)A\right]^{-1/\gamma}$$

$$- \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2\gamma}(1-\gamma)$$

Solve for $(1-\gamma)^{-1/\gamma}A^{-1/\gamma}$

$$\Rightarrow (1-\gamma)^{-1/\gamma}A^{-1/\gamma} = \frac{1}{\gamma}\left[\rho - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right]$$

$$\Rightarrow \boxed{\begin{aligned} c &= \frac{1}{\gamma}\left\{\rho - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right\}W \\ \alpha &= \frac{\mu-r}{\gamma\sigma^2}\end{aligned}}$$

Comments

1.) Portfolio solution is the same as in the static CAPM with quadratic preferences!

2.) Consumption is a fixed fraction of wealth

$\rho \uparrow \Rightarrow c \uparrow$

$\sigma^2 \uparrow \Rightarrow c \downarrow$ if $\gamma > 1$

3.) $c = \rho W$ if $\gamma = 1$

Finite Horizon Problem

$$\max_{c,\alpha} E \int_0^T e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt$$

s.t. 1.) $dW = [(r+\alpha(\mu-r))W - c]dt + \alpha\sigma W \cdot dz$

2.) $V(T) = 0$ } No bequest

HJB

$$-V_t = \max_{c,\alpha}\left\{ e^{-\rho t}\frac{c^{1-\gamma}}{1-\gamma} + [(r+\alpha(\mu-r))W - c]\cdot V_W + \frac{1}{2}\alpha^2\sigma^2 W^2 V_{WW}\right\}$$

FOCs

c: $e^{-\rho t}c^{-\gamma} = V_W \implies c = [e^{\rho t}V_W]^{-1/\gamma}$

α: $(\mu-r)W\cdot V_W + \alpha\sigma^2 W^2 V_{WW} = 0$ } same as before!

Guess

$$V = e^{-\rho t}A(t)^\gamma W^{1-\gamma}$$

$$V_t = -\rho e^{-\rho t}A(t)^\gamma W^{1-\gamma} + \gamma e^{-\rho t}A^{\gamma-1}\dot{A}W^{1-\gamma}$$

$$V_W = (1-\gamma)e^{-\rho t}A(t)^\gamma W^{-\gamma}$$

$$\implies c = (1-\gamma)^{-1/\gamma}A^{-1}W \qquad \frac{-V_W}{W\cdot V_{WW}} = \frac{1}{\gamma}$$

• Sub back into HJB

$$\rho - \gamma A^{-1}\dot{A} = (1-\gamma)^{-1/\gamma} A^{-1} - (1-\gamma)(1-\gamma)^{-1/\gamma} A^{-1} + (1-\gamma)\left[r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right]$$

Multiply by A + collect terms,

$$\dot{A} = \phi A - (1-\gamma)^{-1/\gamma} \qquad \boxed{\phi = \frac{1}{\gamma}\left\{\rho - (1-\gamma)\left[r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]\right\}}$$

Solve this ODE s.t. $A(T) = 0$

Homog. Soln. : $A = B e^{\phi t}$

Particular Soln : $A = \dfrac{(1-\gamma)^{-1/\gamma}}{\phi}$ $\Rightarrow B = -\dfrac{(1-\gamma)^{-1/\gamma}}{\phi} e^{-\phi T}$

Therefore, $A(t) = \dfrac{(1-\gamma)^{-1/\gamma}}{\phi}\left[1 - e^{-\phi(T-t)}\right]$

$$\Rightarrow \boxed{C = \frac{\phi}{1 - e^{-\phi(T-t)}} W} \quad \boxed{\alpha = \frac{\mu - r}{\gamma\sigma^2}}$$

Comments

1.) Same portfolio rule as before! Investor's horizon does not matter.

2.) Horizon does matter for consumption/saving policy. Consume more of your wealth as you get older.