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Department of Economics

Econ 815 | Prof. Kasa
Financial Economics, I | Fall 2016

PROBLEM SET 1 - Dynamic Optimization and State-Contingent Claims
(Solutions)

1. (10 points). **Uncertainty and Exhaustible Resources**. Consider an exhaustible resource that evolves according to the following stochastic differential equation:

$$dR = -xR \cdot dt + \sigma R \cdot dW$$

where R is the current stock of the resource, and x is the current *rate* of extraction. Hence, xR is the current rate of consumption. As usual, dW represents the increment to a Wiener process. This question asks you to study the effects of uncertainty, as parameterized by σ, on the optimal rate of extraction.

Suppose the objective is to maximize the expected discounted utility from the use of the resource,

$$\max_x E\left\{\int_0^\infty e^{-rt}u(xR)dt\right\} \qquad R(0) = R \text{ given}$$

and that preferences take the Constant Relative Risk Aversion form $u(xR) = \frac{(xR)^{1-\gamma}}{1-\gamma}$, where γ is the coefficient of relative risk aversion. For simplicity, suppose the interest rate r is fixed over time.

(a) Write down the (stationary) HJB equation.

If we let $V(R)$ be the value function, the stationary HJB equation becomes

$$rV(R) = \max_x\left\{\frac{(xR)^{1-\gamma}}{1-\gamma} - xRV'(R) + \frac{1}{2}\sigma^2R^2V''(R)\right\}$$

(b) Use a guess-and-verify strategy to solve the HJB equation. (Hint: Try the guess $V(R) = A\frac{R^{1-\gamma}}{1-\gamma}$, where A is an undetermined coefficient).

The first-order condition for x is

$$(xR)^{-\gamma}R - RJ'(R) = 0$$

If we use our guess for $V(R)$ this becomes

$$x^{-\gamma}R^{1-\gamma} = AR^{1-\gamma}$$

Which is easily solved for x,

$$x = A^{-1/\gamma}$$

Substituting this back into the HJB equation gives us

$$r\frac{AR^{1-\gamma}}{1-\gamma} = \frac{A^{1-1/\gamma}R^{1-\gamma}}{1-\gamma} - A^{1-1/\gamma}R^{1-\gamma} - \frac{1}{2}\gamma AR^{1-\gamma}$$

Note that there is a common $R^{1-\gamma}$ term, which can be cancelled out. This leaves us the following equation that determines the unknown constant A

$$r = \gamma A^{-1/\gamma} - \frac{1}{2}\gamma(1-\gamma)\sigma^2$$

(c) Given your answer to part (b), derive the optimal extraction policy x. Does it depend on R? Why or why not? Will the resource ever be depleted?

Since $x = A^{-1/\gamma}$, the previous equation easily implies

$$x = \frac{r}{\gamma} + \frac{1}{2}(1-\gamma)\sigma^2$$

Note that the optimal extraction rate is independent of R. That is, you consume a constant fraction of the remaining stock. This is not a general result. It results from the isoelastic preferences and the particular process specified for R. However, we will see another example of this when we solve the Merton consumption/portfolio problem, where we will find that under the same sort of conditions the optimal policy calls for a constant share of wealth to be invested. Also notice that since x is constant, the process for R becomes a geometric Brownian motion. Such a process can never become negative, and cannot hit zero in finite time. (Because its drift and volatility shrink to zero as $R \to 0$). That is, the resource will never be depleted.

(d) How does x depend on r? Explain. (Hint: Google 'Hotelling's Rule').

The optimal policy for x reveals that the optimal extraction rate increases with the interest rate. In simple deterministic models, Hotelling's Rule implies that the optimal extraction rate should be such that the price of the resource rises at the rate of interest. (The higher the interest rate, the higher the opportunity cost of leaving the resource 'in the ground'). The above result shows that the same sort of considerations apply in our more general stochastic environment.

(e) How does x depend on σ^2? Does greater uncertainty increase or decrease the optimal rate of extraction? How does your answer depend on the value of γ?

Intuitively, one might suspect that greater uncertainty would lower the extraction rate. However, it's not quite that simple. That's because of the lognormal nature of the process, which implies that greater uncertainty also increases the mean growth rate. (Remember Jensen's inequality!). So there are offsetting income and substitution effects at work. From above, we see that the precautionary effect dominates if risk aversion is relatively high ($\gamma > 1$). In this case, greater uncertainty reduces the optimal extraction rate. However, if $\gamma < 1$, the income effect dominates, and a higher variance actually increases the optimal extraction rate. For the knife-edge log utility case ($\gamma = 1$) the optimal extraction rate does not depend on σ^2.

2. (10 points). Consider a world with just two 'states' - Clinton or Trump. There are two firms in the economy - a manufacturer of email servers and a casino. The share price of the server manufacturer is \$34 and the share price of the casino is \$28. Assume their state-contingent profits are as follows:

	Clinton	Trump
Servers	4	1/2
Casino	1	5

(a) What are the (implicit) state-contingent claims prices (ie, the price of \$1 if and only if a given state occurs)?

Let P_C be the Arrow-Debreu price for Clinton, and let P_T be the Arrow-Debreu price for Trump. From the law-of-one-price, we then get

$$\begin{bmatrix} 34 \\ 28 \end{bmatrix} = \begin{bmatrix} 4 & 1/2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} P_C \\ P_T \end{bmatrix}$$

Inverting the matrix, we find the following expression for the AD state-contingent claims prices

$$\begin{bmatrix} P_C \\ P_T \end{bmatrix} = \begin{bmatrix} 4 & 1/2 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 34 \\ 28 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

(b) Given your answer to part (a), what must be the price of a risk-free asset?

A risk-free claim entitles you to a unit of consumption no matter which state occurs. To guarantee this, you must buy AD claims for both states. The price of this will just be the sum of the two AD prices: $P_{riskfree} = P_C + P_T = 12$.