SIMON FRASER UNIVERSITY
Department of Economics

Econ 815 | Prof. Kasa
Financial Economics, I | Fall 2017

MIDTERM EXAM
(Solutions)

Answer the following questions True, False, or Uncertain. Briefly explain your answers. (8 points each).

1. Assets with uncertain payoffs cannot have expected returns below the risk-free interest rate.

   FALSE. *According to the CAPM, an asset's expected return depends on its covariance with the market. An asset that covaries negatively with the market will have an expected return below the risk-free rate. It's benefits in terms of diversification and portfolio risk reduction compensate for its low return.*

2. If two individuals are unwilling to undertake a project on their own, they will also be unwilling to do it on a 50/50 basis, where each individual puts up half the money and gets half the return.

   FALSE. *Although dividing the project in half causes your returns to be cut in half, it causes your risk to fall by even more, since risk/variance scales with the square of the amount invested. In fact, a risk averse agent will always accept a sufficiently small stake in a favorable bet (ie, a bet with a positive expected return).*

3. According to the Merton model, old people should hold less of their wealth in stocks.

   FALSE. *The benchmark Merton model (with CRRA utility and iid lognormal returns) implies that portfolio shares are constant over time. However, it is possible to extend the model to produce portfolio shares that depend on wealth. In class, we discussed how nontradeable labor income might cause old people to invest less in the stock market (because they don't have much labor income left to supply!).*

4. A risk averse person will always refuse a fair bet. (A fair bet has an expected payoff of zero).

   TRUE. *The reason is basically the same as in question 2. Refusal to accept an actuarial fair bet defines risk aversion.*

5. If markets are complete, then expected returns on all assets are the same.

   FALSE. *The defining feature of complete markets is that all individuals have the same marginal rates of substitution across dates and states. When this is true, there are no longer any gains from trade. However, this can hold true even if assets offer wildly different returns across dates and states.*

   The following questions are short answer. Briefly explain your answer. Clarity will be rewarded.

6. (15 points). What does the CAPM predict about the relationship between stock prices and the (riskless) interest rate? Assuming expected future stock prices are unaffected by the interest rate change, how are current prices affected? Does it depend on a stock's β? (Hint: What is the relationship between expected returns and current prices?)

   According to the CAPM,

$$R_i = R_f + \beta_i(R_m - R_f) \qquad \Rightarrow \qquad R_i = (1-\beta_i)R_f + \beta_i R_m$$

   Conditional on R_m, notice that if $\beta_i > 1$ then $cov(R_i, R_f) < 0$, whereas if $\beta_i < 1$ we have $cov(R_i, R_f) > 0$. Next, remember that if future payoffs/prices are held constant, then current prices and expected returns move inversely. When returns go up, prices go down, and vice versa. (This is the way discount bonds, like

Treasury bills, work). Therefore, we can see that if we condition on the level of the market itself (ie, hold R_m constant), then CAPM predicts that prices of stocks with β's above one will go up when interest rates go up, whereas the prices of stocks with β's below one will move inversely with interest rates. (I actually don't know whether anyone has checked this in the data!

7. (15 points). Suppose the expected return on the market portfolio is 7%, and the risk-free rate is 1%. The standard deviation of the market portfolio is 16%. Assuming the CAPM holds,

   (a) What is the equation of the "Capital Market Line"?

   The Capital Market Line gives the relationship between risk and return on all efficient portfolios. It is given by the equation

$$r = r_f + \left(\frac{r_m - r_f}{\sigma_m}\right)\sigma$$

   (b) If you desire a 4% expected return, what will be the associated standard deviation of this position (assuming it's efficient)? If you have \$1000 to invest, how should you allocate it to achieve this position?

   Plugging in the given information, we have

$$.04 = .01 + \left(\frac{.07 - .01}{.16}\right)\sigma \qquad \Rightarrow \qquad \sigma = (.07 - .01)(.16/.06) = .08$$

   Let α be the share invested in the market portfolio. To achieve an expected portfolio return of 4%, we must pick α so that $.04 = \alpha(.07) + (1-\alpha)(.01)$. This implies $\alpha = 0.5$, so you should invest \$500 in the market and \$500 in the risk-free asset.

8. (30 points). Suppose an individual has the utility function

$$U = \max_{c} E \int_0^\infty \frac{1}{1-\gamma} c^{1-\gamma} e^{-\rho t} dt$$

As usual, suppose the individual can invest in a risk-free asset and/or a risky asset, with prices that follow the processes

$$\begin{aligned} \frac{dB}{B} &= rdt \\ \frac{dP}{P} &= \mu dt + \sigma dz \end{aligned}$$

where dz is an increment to a Wiener process.

   (a) Let w be the agent's current wealth, and α be the share of his wealth invested in the risky asset. Write down the agent's budget constraint.

$$dw = [(r + \alpha(\mu - r))w - c]dt + \alpha\sigma dB$$

   (b) Write down the stationary Hamilton-Jacobi-Bellman equation.

$$\rho V(w) = \max_{c,\alpha} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} + [(r + \alpha(\mu - r))w - c]V'(w) + \frac{1}{2}\alpha^2\sigma^2 V''(w) \right\}$$

   (c) Prove that $V(w) = \frac{A}{1-\gamma} w^{1-\gamma}$ solves the HJB equation, where A is a constant.

   The FOCs for c and α are

$$\begin{aligned} c^{-\gamma} - V'(w) &= 0 \\ (\mu - r)wV'(w) + \alpha w^2 V''(w) &= 0 \end{aligned}$$

   Using the given guess, we have $V'(w) = Aw^{-\gamma}$ and $V''(w) = -\gamma A w^{-\gamma-1}$. Substitute these into the FOCs, solve for c and α, and then substitute back into the HJB equation. You will see that w cancels out! You could proceed to solve for A is you want, but it is not necessary in this case.

(d) Given the above value function, derive the agent's optimal portfolio, and interpret it economically.

Solving the FOC for α and using our guess for $V(w)$ gives

$$\alpha = \frac{(\mu - r)}{\gamma\sigma^2}$$

Hence, the share invested in the risky asset increases with the excess return, but decreases with the coefficient of relative risk aversion and the volatility of the risky asset. This is exactly what you would expect, intuitively, but it's useful to have an explicit formula that allows us to quantify the effects.