3.) **Current Account Autarky.** No intertemporal trade, but can exchange equal-valued claims across states next period.

Optimal 2nd-period consumption is now

\[ P(s)C_2(s) = \Pi(s)[P(1)Y_2(1) + P(2)Y_2(2)] \]

Using this in the Euler eq.,

\[ 1 + r^{ca} = \frac{U'(Y_1)}{\beta \sum \Pi(s) U'\left[ \Pi(s)\left( P(1)Y_2(1) + P(2)Y_2(2) \right) / P(s) \right]} \]

The CA will be balanced if \( r^{ca} = r \). In other words, when \( r^{ca} = r \), net capital flows are zero. However, there can still be large two-way, or gross, capital flows.

A country's demand for state-contingent claims can now be written:

\[ B_a(1) = C_a(1) - Y_a(1) = \frac{P(2)}{P(1)} \Pi(1) Y_a(1) \left[ \frac{P(1)^{A}}{P(1)^{A}} - \frac{P(1)}{P(2)} \right] \]

\[ B_a(2) = C_a(2) - Y_a(2) = -\Pi(2) Y_a(1) \left[ \frac{P(1)^{A}}{P(1)^{A}} - \frac{P(1)}{P(2)} \right] \]

A country imports the asset with the relatively high autarky price, while exporting the one with the relatively low autarky price. This reflects pure smoothing across states.
A 2-Country Arrow-Debreu Model

Now AD state prices become endogenous.

Basic Idea: Combine FOCs with global market clearing.

**Prices/FOCs**

\[
\frac{P(s)}{1+r} u'(c_t) = \pi(s) \beta u'(c_{a}(s))
\]

\[
\frac{P(s)}{1+r} u'(c^*_t) = \pi(s) \beta u'(c^*_a(s))
\]

**Market-Clearing**

\[
c_t + c^*_t = Y_t + Y^*_t = Y^*_w
\]

\[
c_s(s) + c^*_s(s) = Y_s(s) + Y^*_s(s) = Y^*_w(s) \quad s = 1, 2, \ldots, S
\]

Suppose identical CRRA preferences, \( u(c) = \frac{1}{1-\rho} c^{1-\rho} \)

\[
c^*_2(s) = \left[ \frac{\beta \pi(s)(1+r)}{P(s)} \right]^{\rho} c_t
\]

\[
c^*_2(s) = \left[ \frac{\beta \pi(s)(1+r)}{P(s)} \right]^{\rho} c^*_t
\]

Add and impose market-clearing,

\[
Y^*_w(s) = \left[ \frac{\beta \pi(s)(1+r)}{P(s)} \right]^{\rho} Y^*_w
\]

\[\Rightarrow \quad \frac{P(s)}{1+r} = \beta \pi(s) \left[ \frac{Y^*_w(s)}{Y^*_w} \right]^{\rho}\]
Take ratios across states,

\[
P(s) = \frac{\pi(s)}{\pi(s')} \left[ \frac{Y^w(s)}{Y^w(s')} \right]^{-\rho}
\]

Note,
1.) Full insurance not feasible with aggregate uncertainty
2.) States where aggregate output is relatively abundant have low AD prices.

Using the no arbitrage condition, \( \Sigma P(s) = 1 \), we can solve separately for \( P(s) \) and \( 1 + \rho \)

\[
P(s') = 1 - \sum_{s \neq s'} P(s) = 1 - P(s') \sum_{s \neq s'} \frac{P(s)}{P(s')}
\]

\[
\Rightarrow P(s') = \frac{\pi(s') [Y^w(s')]^{-\rho}}{\sum_{s \neq s'} \pi(s) [Y^w(s)]^{-\rho}}
\]

\[
1 + \rho = \frac{[Y^w]^{-\rho}}{\beta \sum_{s \neq s'} \pi(s) [Y^w(s)]^{-\rho}}
\]

As usual, higher current output lowers real interest rates, while higher future output (in any state) raises real interest rates.
Complete markets implies strong (and testable) restrictions on international consumption data.

**Complete mkt. \implies** Equal MRS across countries (state-by-state and date-by-date)

**Intertemporal**

1) \[ \frac{\Pi(s) \beta u'(c_2(s))}{u'(c_1)} = \frac{\Pi(s)}{1+r} = \frac{\Pi(s) \beta u'(c_2^*(s))}{u'(c_1^*)} \]

**Across States**

2) \[ \frac{\Pi(s) u'(c_2(s))}{\Pi(s') u'(c_2(s'))} = \frac{\Pi(s)}{\Pi(s')} = \frac{\Pi(s) u'(c_2^*(s))}{\Pi(s') u'(c_2^*(s'))} \]

Suppose \( U = U^* = \frac{1}{1-c} c^{1-c} \). The (2.) implies

\[ \frac{c_2(s)}{c_2(s')} = \frac{y_2^w(s)}{y_2^w(s')} = \frac{c_2^*(s)}{c_2^*(s')} \]

And (1.) implies

\[ \frac{c_2(s)}{c_1} = \frac{y_2^w(s)}{y_1^w} = \frac{c_2^*(s)}{c_1^*} \]
Note that (3) implies,
\[
\frac{C_2(s)}{Y_2(s)} = \frac{C_2(s')}{Y_2(s')} \quad \text{and} \quad \frac{C_2^*(s)}{Y_2^*(s)} = \frac{C_2^*(s')}{Y_2^*(s')}
\] \text{constant shares across states}

And from (4)
\[
\frac{C_1}{Y_1} = \frac{C_2(s)}{Y_2(s)} \quad \text{and} \quad \frac{C_1^*}{Y_1^*} = \frac{C_2^*(s)}{Y_2^*(s)}
\] \text{constant shares across dates}

Suppose \( \frac{C}{Y} = \mu \) and \( \frac{C^*}{Y^*} = 1 - \mu \)

Then \( \mu \) is constant, and determined by individual budget constraints.

\( \mu \) = Share of Country H's wealth evaluated at the equilibrium AD prices.

Empirically, if we assume CRRA and differentials the Euler eqs. we get the testable equation:
\[
\log \left[ \frac{C_2^*(s)}{C_1^*} \right] = \left( \frac{P_m}{P_n} \right) \log \left[ \frac{C_2(s)}{C_1}\right] + \frac{1}{P_n} \log \left( \frac{P_n}{P_m}\right)
\]

Note, individual country consumption can vary, as a reflection of aggregate risk and differential risk aversion, but cross-country consumptions should be perfectly correlated.
<table>
<thead>
<tr>
<th>Country</th>
<th>corr(C,Cw)</th>
<th>corr(y,yw)</th>
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<tr>
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<td>.52</td>
</tr>
<tr>
<td>Developing Country Avg.</td>
<td>- .10</td>
<td>.05</td>
</tr>
</tbody>
</table>

Hence, this prediction of complete markets is rejected
Risk-Sharing

With complete markets, the 2nd Welfare Theorem applies, and we can interpret a competitive equilibrium as the outcome of Pareto (or social planning) problem. This is a useful perspective when studying risk sharing.

Risk-sharing is usually analyzed with HARA utility functions, since they deliver linear sharing rules. [Wilson (1968) - "The Theory of Syndicates"]

\[
\text{HARA: } U(c) = \alpha \frac{c}{1-\gamma} \left( \beta + \frac{c}{1-\gamma} \right)^{1-\gamma}
\]

1.) \( \lim_{\gamma \to \infty} \Rightarrow U(c) = -\beta e^{-\gamma p} \)  
\( \beta \geq 0 \)  
\( \gamma < 1 \)  
\( \gamma \geq 1 \)

2.) \( \beta = 0 \Rightarrow U(c) = \frac{c}{1-\gamma} \)  
\( \beta \geq 0 \)  
\( \gamma < 1 \)  
\( \gamma > 1 \)

3.) \( \gamma = 1 \Rightarrow U(c) = -\frac{1}{2} (\beta - c)^2 \)  
\( \beta \geq c \)  
\( \beta < c \)

The HARA class is useful since it features linear risk tolerance \([\text{Risk Tolerance} = -\frac{U'(c)}{U''(c)}]\)
Examples of Sharing Rules

Define \( Y^w(s) = \sum_{i=1}^{N} y_i(s) \)

\[ \text{Total world endowment in State } s \text{ (} N \text{ countries)} \]

\[ \text{Pareto Weights} \]

\[ \text{Pareto Problem: } \max_{C_i(s)} \sum_{i=1}^{N} \lambda_i \{ \sum_{s \in S} \pi(s) U(C_i(s)) \} \]
subject to \( \sum_{i=1}^{N} C_i(s) = Y^w(s) \) for all states

1) 2 countries, Identical exponential, \( u = -\frac{1}{2} e^{-c} \)

\[ C_1(s) = \frac{1}{2} Y^w(s) + \frac{1}{2} \frac{1}{2} \log \left( \frac{\lambda_1}{\lambda_2} \right) \]
\[ C_2(s) = \frac{1}{2} Y^w(s) - \frac{1}{2} \frac{1}{2} \log \left( \frac{\lambda_1}{\lambda_2} \right) \]

2) Identical CRRA, \( u = \frac{1}{1-c} C^{1-c} \)

\[ C_1(s) = \varphi Y^w(s) \]
\[ C_2(s) = (1 - \varphi) Y^w(s) \]

where \( \varphi = \frac{\lambda_1 \nu e}{\lambda_1 \nu e + \lambda_2 \nu e} \)
Adding Production + Investment

- So far, we have only considered endowment economies. Now let's assume 2nd-period output is produced by competitive firms. The important new feature now is that investment returns are risky, i.e., they are state-contingent.

- Assume home & foreign production functions are:
  \[ Y_a(s) = A(s)F(K_a) \quad Y_x(s) = A^*(s)F(K_x^*) \]

- For simplicity, suppose \( K_1 = 0 \), so \( I = K_2 \)

- As usual, firms maximize the PDV of profits, but note that now profits are valued at AD prices.

Objective Function

\[
\max_{K_2} \sum_{s=1}^{S} \frac{\rho(s)}{1+r} \left[ A(s)F(K_a) + K_e \right] - K_2
\]

First-Order Conditions

\[
\sum_{s=1}^{S} \frac{\rho(s)}{1+r} \left[ A(s)F'(K_a) + 1 \right] = 1
\]

\[
\sum \rho(s)A(s)F'(K_a) = r = \sum \rho(s)A^*(s)F''(K_x^*)
\]
From Euler,

\[ U'(c_i) = \sum_{s} \pi(s) \beta U'(c_{s|i}) [A(s)F'(K_s) + 1] \]

\[ = \sum_{s} \pi(s) \beta U'(c_{z|i}) [A(s)F''(K_z) + 1] \]

Individuals are indifferent between investing at home or in foreign.

- Same risk-sharing implications as before, but with world output defined as net of the efficient investment decisions.

- Note that in this complete markets environment, the ownership of firms is completely irrelevant. Firms make the same investment decisions regardless of who owns them. That's because H and F residents have exactly the same cross-date and cross-state MRS.