3.) Current Account Autarky. No intertemporal trade, but can exchange equal-valued claims across states next period.

Optimal 2^{nd}-period consumption is now

$$P(s)C_2(s) = \pi(s)\left[P(1)Y_2(1) + P(2)Y_2(2)\right]$$

Using this in the Euler eq.,

$$1+r^{CA} = \frac{u'(Y_1)}{\beta \sum \pi(s)\, u'\left[\pi(s)\left(P(1)Y_2(1)+P(2)Y_2(2)\right)/P(s)\right]}$$

The CA will be balanced if $r^{CA} = r$. In other words, when $r^{CA} = r$, net capital flows are zero. However, there can still be large two-way, or gross, capital flows.

A country's demand for state-contingent claims can now be written:

$$B_2(1) = C_2(1) - Y_2(1) = \frac{P(2)}{P(1)}\pi(2)Y_2(1)\left[\frac{P(1)^A}{P(2)^A} - \frac{P(1)}{P(2)}\right]$$

$$B_2(2) = C_2(2) - Y_2(2) = -\pi(2)Y_2(1)\left[\frac{P(1)^A}{P(2)^A} - \frac{P(1)}{P(2)}\right]$$

A country imports the asset with the relatively high autarky price, while exporting the one with the relatively low autarky price. This reflects pure smoothing across states.

A 2-Country Arrow-Debreu Model

Now AD state prices become endogenous

Basic Idea: Combine FOCs with global market clearing

Prices / FOCs

$$\frac{P(s)}{1+r} u'(c_1) = \pi(s) \beta u'(c_2(s))$$

$$\frac{P(s)}{1+r} u'(c_1^*) = \pi(s) \beta u'(c_2^*(s))$$

Market-Clearing

$$c_1 + c_1^* = Y_1 + Y_1^* = Y_1^w$$

$$c_2(s) + c_2^*(s) = Y_2(s) + Y_2^*(s) = Y_2^w(s) \qquad s = 1, 2, \cdots S$$

Suppose identical CRRA preferences, $u(\cdot) = \frac{1}{1-\rho} c^{1-\rho}$

$$c_2(s) = \left[\frac{\beta \pi(s)(1+r)}{P(s)} \right]^{1/\rho} c_1$$

$$c_2^*(s) = \left[\frac{\beta \pi(s)(1+r)}{P(s)} \right]^{1/\rho} c_1^*$$

Add + impose market-clearing,

$$Y_2^w(s) = \left[\frac{\beta \pi(s)(1+r)}{P(s)} \right]^{1/\rho} Y_1^w$$

$$\Longrightarrow \boxed{\frac{P(s)}{1+r} = \beta \pi(s) \left[\frac{Y_2^w(s)}{Y_1^w} \right]^{-\rho}}$$

Take ratios across states,

$$\frac{P(s)}{P(s')} = \frac{\pi(s)}{\pi(s')} \left[\frac{Y_2^w(s)}{Y_2^w(s')} \right]^{-\rho}$$

Note,

1.) Full insurance not feasible with aggregate uncertainty

2.) States where aggregate output is relatively abundant have low AD prices.

Using the no arbitrage condition, $\sum P(s) = 1$, we can solve separately for $P(s)$ and $1+r$

$$P(s') = 1 - \sum_{s \neq s'} P(s) = 1 - P(s') \sum_{s \neq s'} \frac{P(s)}{P(s')}$$

$$\Rightarrow \boxed{\begin{aligned} P(s') &= \frac{\pi(s')\left[Y_2^w(s')\right]^{-\rho}}{\sum_{s=1}^{S} \pi(s)\left[Y_2^w(s)\right]^{-\rho}} \\ 1+r &= \frac{\left[Y_1^w\right]^{-\rho}}{\beta \sum_{s=1}^{S} \pi(s)\left[Y_2^w(s)\right]^{-\rho}} \end{aligned}}$$

As usual, higher current output lowers real interest rates, while higher future output (in any state) raises real interest rates.

Complete markets implies strong (and testable) restrictions on international consumption data.

Complete mkts. $\Longrightarrow$ Equal MRS across countries (state-by-state and date-by-date)

Intertemporal

1.) $$\frac{\pi(s)\beta u'(c_2(s))}{u'(c_1)} = \frac{p(s)}{1+r} = \frac{\pi(s)\beta u'(c_2^*(s))}{u'(c_1^*)}$$

Across States

2.) $$\frac{\pi(s)u'(c_2(s))}{\pi(s')u'(c_2(s'))} = \frac{p(s)}{p(s')} = \frac{\pi(s)u'(c_2^*(s))}{\pi(s')u'(c_2^*(s'))}$$

Suppose $u = u^* = \frac{1}{1-\rho} c^{1-\rho}$. The (2.) implies

(3.) $$\frac{c_2(s)}{c_2(s')} = \frac{Y_2^w(s)}{Y_2^w(s')} = \frac{c_2^*(s)}{c_2^*(s')}$$

And (1.) implies

(4.) $$\frac{c_2(s)}{c_1} = \frac{Y_2^w(s)}{Y_1^w} = \frac{c_2^*(s)}{c_1^*}$$

Note that (3) implies,

$$\frac{c_2(s)}{Y_2^w(s)} = \frac{c_2(s')}{Y_2^w(s')} \quad \text{and} \quad \frac{c_2^*(s)}{Y_2^w(s)} = \frac{c_2^*(s')}{Y_2^w(s')}$$ } constant shares across states

And from (4)

$$\frac{c_1}{Y_1^w} = \frac{c_2(s)}{Y_2^w(s)} \quad \text{and} \quad \frac{c_1^*}{Y_1^w} = \frac{c_2^*(s)}{Y_2^w(s)}$$ } constant shares across dates

Suppose $\frac{c}{y^w} = \mu$ and $\frac{c^*}{y^w} = 1-\mu$

Then μ is constant, and determined by individual budget constraints.

μ = Share of Country H's wealth evaluated at the equilibrium AD prices.

Empirically, if we assume CRRA and differentiable the Euler eqs. we get the testable equation:

$$\boxed{\log\left[\frac{c_2^n(s)}{c_1^n}\right] = \left(\frac{\rho_m}{\rho_n}\right)\log\left[\frac{c_2^m(s)}{c_1^m}\right] + \frac{1}{\rho_n}\log\left(\frac{\beta_n}{\beta_m}\right)}$$ for all country pairs m, n

Note, individual country consumption can vary, as a reflection of aggregate risk and differential risk aversion, but cross-country consumptions should be <u>perfectly correlated</u>

Empirical Evidence : 1973-1992

Penn World Tables

	$corr(c, c^w)$	$Corr(y, y^w)$
Canada	.56	.70
USA	.52	.68
UK	.63	.62
France	.45	.60
Germany	.63	.70
Italy	.27	.51
Japan	.38	.46
OECD Avg.	.43	.52
Developing Country Avg.	-.10	.05

Hence, this prediction of complete markets is rejected

Risk-Sharing

With complete markets, the 2nd Welfare Theorem applies, and we can interpret a competitive equilibrium as the outcome of Pareto (or social planning) problem. This is a useful perspective when studying risk sharing.

Risk-sharing is usually analyzed with HARA utility functions, since they deliver linear sharing rules. [Wilson (1968) - "The Theory of Syndicates"]

Hyperbolic Absolute Risk Aversion

$$\boxed{\underline{\text{HARA}}:\ u(c) = \alpha \frac{\gamma}{1-\gamma}\left(\beta + \frac{c}{\gamma}\right)^{1-\gamma}}$$

1.) $\lim_{\gamma \to \infty} \Rightarrow u(c) = -\beta e^{-c/\beta}$ } constant absolute risk aversion

2.) $\beta = 0 \Rightarrow u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ } constant relative risk aversion

3.) $\gamma = -1 \Rightarrow u(c) = -\frac{1}{2}(\beta - c)^2 \quad \beta > c$

The HARA class is useful since it features linear risk tolerance [Risk Tolerance $= -\frac{u'}{u''}$]

Examples of Sharing Rules

Define $Y^w(s) = \sum_{i=1}^{N} y_i(s)$ } Total world endowment in State s (N countries)

Pareto weights

~~1) 2 countries~~ **Pareto Problem**: $\max\limits_{c_i(s)} \sum_{i=1}^{N} \lambda_i \left\{ \sum_{s=1}^{S} \pi(s) u(c_i(s)) \right\}$

subject to $\sum_{i=1}^{N} c_i(s) = Y^w(s) \quad \forall s$

1.) 2 countries, Identical exponential, $u = -\frac{1}{\sigma} e^{-\sigma c}$

$$c_1(s) = \frac{1}{2} Y^w(s) + \frac{1}{\sigma}\frac{1}{2}\log\left(\frac{\lambda_1}{\lambda_2}\right)$$

$$c_2(s) = \frac{1}{2} Y^w(s) - \frac{1}{\sigma}\frac{1}{2}\log\left(\frac{\lambda_1}{\lambda_2}\right)$$

2.) Identical CRRA, $u = \frac{1}{1-\rho} c^{1-\rho}$

$$c_1(s) = \phi Y^w(s)$$

$$c_2(s) = (1-\phi) Y^w(s)$$

where $\phi = \dfrac{\lambda_1^{1/\rho}}{\lambda_1^{1/\rho} + \lambda_2^{1/\rho}}$

Adding Production + Investment

- So far, we have only considered endowment economies. Now let's assume 2^{nd}-period output is produced by competitive firms. The important new feature now is that investment returns are risky, i.e., they are state-contingent.

- Assume Home + Foreign production functions are:

$$Y_2(s) = A(s)F(K_2) \qquad Y_2^*(s) = A^*(s)F(K_2^*)$$

- For simplicity, suppose $K_1 = 0$, so $I = K_2$

- As usual, firms maximize the PDV of profits, but note that now profits are valued at AD prices.

Objective Function

$$\max_{K_2} \sum_{s=1}^{S} \frac{P(s)}{1+r}\left[A(s)F(K_2) + K_2\right] - K_2$$

First-Order Conditions

Note, separation between investment and (country-specific) consumption

$$\sum_{s=1}^{S} \frac{P(s)}{1+r}\left[A(s)F'(K_2) + 1\right] = 1$$

Efficient International Allocation of Investment

$$\Rightarrow \boxed{\sum P(s)A(s)F'(K_2) = r = \sum P(s)A^*(s)F^{*\prime}(K_2^*)}$$

From Euler,

$$u'(c_1) = \sum_{s=1}^{S} \pi(s)\beta u'(c_2(s))\left[A(s)F'(K_2)+1\right]$$

$$= \sum_{s=1}^{S} \pi(s)\beta u'(c_2(s))\left[A^*(s)F^{*\prime}(K_2^*)+1\right]$$

Individuals are indifferent between investing at Home or in Foreign.

- Same risk-sharing implications as before, but with world output defined as <u>net</u> of the efficient investment decisions.

- Note that in this complete markets environment, the ownership of firms is completely irrelevant. Firms make the same investment decisions regardless of who owns them. That's because H and F residents have exactly the same cross-date + cross-state MRS!