PROBLEM SET 3
(Due March 25)

1. (30 points). Consider a 2-period small open endowment economy facing the exogenous world interest rate \( r \) on riskless loans. Date 1 output is \( Y_1 \). There are \( S \) states of nature on date 2 that differ according to the output realizations \( Y_2(s) \). The probability that state \( s \) is realized is known to be \( \pi(s) \). The representative domestic household maximizes the following expected lifetime utility function:

\[
U_1 = C_1 - \frac{a}{2} (C_1)^2 + \beta E \left[ C_2 - \frac{a}{2} (C_2)^2 \right] \quad a > 0
\]

Assume that the rate of time preference equals the interest rate, so that \( \beta(1 + r) = 1 \). When markets are incomplete the household faces the sequence of budget constraints

\[
B_2 = (1 + r)B_1 + Y_1 - C_1
\]
\[
C_2(s) = (1 + r)B_2 + Y_2(s) \quad s = 1, 2, \ldots, S
\]

where \( B_i \) denotes net foreign assets at the beginning of period-\( i \). Assume that the parameters are such that the marginal utility of consumption, \( 1 - aC \), is always positive.

(a) Start by temporarily ignoring the nonnegativity constraints \( C_2(s) \geq 0 \) on date 2 consumption. Compute optimal date 1 consumption, \( C_1 \). What are the implied values of \( C_2(s) \)? What do you think your answer would be with an infinite horizon and output uncertainty in each future period? (Hint: Remember chapter 2!).

(b) Now let’s worry about the nonnegativity constraint on \( C_2(s) \). Without loss of generality, renumber the date 2 states so that \( Y_2(1) = \min_s [Y_2(s)] \). Show that if

\[
(1 + r)B_1 + Y_1 + \frac{2 + r}{1 + r} Y_2(1) \geq E_1 Y_2
\]

then the \( C_1 \) computed in part (a) (for the 2-period case) is still valid. What is the intuition? Suppose the preceding inequality doesn’t hold. Show that the optimal date 1 consumption is lower (reflecting a precautionary savings effect) and equals

\[
C_1 = (1 + r)B_1 + Y_1 + \frac{Y_2(1)}{1 + r}
\]

(Hint: Apply Kuhn-Tucker). What is the intuition here? Does the usual Euler equation hold in this case?
(c) Now assume the household has access to complete Arrow-Debreu markets, with $p(s)$ being the exogenous state $s$ Arrow-Debreu contingent claims price for state $s$. Assume these prices are actuarial fair, so that $p(s) = \pi(s)$. Compute the optimal values of $C_1$ and $C_2(s)$ in this case. Why can we ignore nonnegativity constraints in this complete markets case?

2. (30 points). Consider a two-country, one-good world where agents in each country have preferences

$$U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\rho}}{1-\rho}$$

Country-1’s endowment is $y_{1t} = 1$ for all $t$. Country-2’s endowment is $y_{2t} = \gamma^t$, where $\gamma > 1$.

(a) Describe the competitive equilibrium with complete markets. (Hint: Consider the Pareto problem).

(b) Now suppose agents cannot commit to their Arrow-Debreu contracts, and can go live under autarky at any time. Derive each agent’s participation constraints (for each $t$).

(c) Does the complete markets allocation in part (a) satisfy the participation constraints? If not, what is the constrained-optimal allocation?

3. (40 points). This question is about the trade balance and the terms of trade in open-economy RBC models. Consider a world consisting of two exchange economies, Country 1 and Country 2. Country 1 receives a stochastic endowment sequence of “apples”, $a_t(s^t)$, and Country 2 receives a stochastic endowment of “bananas”, $b_t(s^t)$, where the notation $s^t$ represents the fact that endowments depend on the history of states realized up to period-$t$. Residents of both countries have the same preferences

$$U(a, b) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t)[a_t(s^t)^{1-\rho} + b_t(s^t)^{1-\rho}] / (1 - \rho)$$

where $\pi(s^t)$ represents the probability of history $s^t$ (so that this is just expected utility).

(a) Compute the Pareto optimal allocation, and describe the supporting prices.

(b) Let $q$ be a country’s terms of trade, defined as the the relative price of its imports (so that an decrease in $q$ represents a terms of trade improvement). Compute $q$ for country 1.

(c) Derive an expression for country 1’s trade balance, $nx_{1,t} = a_t - q_t b_t$.

(d) What is the relationship between $nx_{1,t}/y_{1t}$ and $q_t$, where $y_{1t}$ is country 1’s GDP? What is the relationship between $nx_{1,t}/y_{1t}$ and $y_{1t}$? Are these consistent with the data?