7

History-Dependent Public Policies

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7.1 Introduction

For the purpose of making some general points about history-dependent public policies and their representations, we study a model in which a benevolent tax authority is forced to raise a prescribed present value of revenues by imposing a distorting flat rate tax on the output of a competitive representative firm that faces costs of adjusting its output. That the firm lives within a rational expectations equilibrium imposes restrictions on the tax authority.[1]

We compare two timing protocols. In the first, an infinitely lived benevolent tax authority solves a Ramsey problem. This means that the authority chooses a sequence of tax rates once-and-for-all at time 0. In the second timing protocol, there is a sequence of tax authorities, each choosing only a time t tax rate. Under both timing protocols, optimal tax policies are history-dependent. But the history dependence reflects different economic forces across the two timing protocols. In the first, history dependence expresses the time-inconsistency of the Ramsey plan. In the second, it reflects the unfolding of constraints that assure that at a time t government wants to confirm the representative firm's expectations about government actions. We discuss recursive representations of history-dependent tax policies under both timing protocols.

The first timing protocol models a policy-maker who can be said to 'commit'. To obtain a recursive representation of a Ramsey policy, we compare two methods. We first apply a method proposed by Kydland and Prescott (1980) that uses a promised marginal utility to augment authentic state variables. We then apply a closely related method of Miller and Salmon (1985), Pearlman et al. (1986), and Backus and Driffill (1986). This method uses a 'co-state on a co-state' variable to augment the authentic state variables. After applying both methods, we describe links between them and confirm that they recover the same Ramsey plan.

Turning to the second timing protocol in which the tax rate is chosen sequentially, we use the notion of a sustainable plan proposed by Chari and Kehoe (1990), also referred to as a credible public policy by Stokey (1989). A key idea here is that history-dependent policies can be arranged so that, when regarded as a representative firm's forecasting functions, they confront policy-makers with incentives to confirm them. We follow Chang (1998) in expressing such history-dependent plans recursively. Credibility considerations contribute an additional auxiliary state variable (above and beyond the auxiliary state variable appearing in the first timing protocol). This new state variable is a promised value to the planner. It expresses how things must unfold to give the government the incentive to confirm private sector expectations when the government chooses sequentially.

We write this chapter partly because we observe occasional confusions about the consequences of our two timing protocols and about recursive representations of government policies under them. It is erroneous to regard a recursive representation of the Ramsey plan as in any way 'solving' a time-inconsistency problem. In contrast, the evolution of the auxiliary state variable that augments the authentic ones under our first timing protocol ought to be viewed as *expressing* the time inconsistency of a Ramsey plan. Despite that, in literatures about practical monetary policy one frequently sees efforts to 'sell' Ramsey plans in settings where our second, sequential timing protocol more accurately characterizes decision-making. One of our purposes is to issue a warning to beware of discussions of credibility if you don't see recursive representations of policies with the complete list of state variables appearing in the Chang (1998)-like analysis of Section 7.9 below.

7.2 Rational expectations equilibrium

A representative competitive firm sells output q_t for price p_t, where market-wide output is Q_t. The market as a whole faces a downward sloping inverse demand function

$$p_t = A_0 - A_1 Q_t, \quad A_0 > 0, A_1 > 0. \tag{7.1}$$

* We thank Marco Bassetto for very helpful comments.

[1] We could also call a competitive equilibrium a rational expectations equilibrium.

The representative firm has given initial condition q_0, endures quadratic adjustment costs $\frac{d}{2}(q_{t+1}-q_t)^2$, and pays a flat rate tax τ_t per unit of output. The firm faces what it regards as exogenous sequences $\{p_t, \tau_t\}_{t=0}^{\infty}$ and chooses $\{q_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{p_t q_t - \frac{d}{2}(q_{t+1}-q_t)^2 - \tau_t q_t\right\}. \tag{7.2}$$

Let $u_t = q_{t+1} - q_t$ be the firm's 'control' variable at time t. First-order conditions for the firm's problem are

$$u_t = \frac{\beta}{d} p_{t+1} + \beta u_{t+1} - \frac{\beta}{d}\tau_{t+1} \tag{7.3}$$

for $t \geq 0$.

Notation. For any scalar x_t, let $\vec{x} = \{x_t\}_{t=0}^{\infty}$.

To compute a rational expectations equilibrium, it is appropriate to take (7.3), eliminate p_t in favour of Q_t by using (7.1), and then set $q_t = Q_t$, thereby making the representative firm representative.[2] We arrive at

$$u_t = \frac{\beta}{d}\left[A_0 - A_1 Q_{t+1}\right] + \beta u_{t+1} - \frac{\beta}{d}\tau_{t+1}. \tag{7.4}$$

We also have

$$Q_{t+1} = Q_t + u_t. \tag{7.5}$$

Equations (7.1), (7.4), and (7.5) summarize competitive equilibrium sequences for $(\vec{p}, \vec{Q}, \vec{u})$ as functions of the path $\{\tau_{t+1}\}_{t=0}^{\infty}$ for the flat rate distorting tax τ.

Definition 7.2.1 *Given a tax sequence $\{\tau_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium is a price sequence $\{p_t\}_{t=0}^{\infty}$ and an output sequence $\{Q_t\}_{t=0}^{\infty}$ that satisfy (7.1), (7.4), and (7.5).*

Definition 7.2.2 *For any sequence $\vec{x} = \{x_t\}_{t=0}^{\infty}$, $\vec{x}_1 = \{x_t\}_{t=1}^{\infty}$ is called a* continuation *sequence or simply a* continuation.

Remark 7.2.3 *A competitive equilibrium consists of a first period value $u_0 = Q_1 - Q_0$ and a continuation competitive equilibrium with initial condition Q_1. A continuation of a competitive equilibrium is a competitive equilibrium.*

Following the lead of Chang (1998), we shall make extensive use of the following property:

[2] It is important not to set $q_t = Q_t$ prematurely. To make the firm a price taker, this equality should be imposed *after* and not *before* solving the firm's optimization problem.

Remark 7.2.4 *A continuation $\vec{\tau}_1 = \{\tau_{t+1}\}_{t=1}^{\infty}$ of a tax policy $\vec{\tau}$ influences u_0 via (7.4) entirely through its impact on u_1. A continuation competitive equilibrium can be indexed by a u_1 that satisfies (7.4).*

Definition 7.2.5 *With some abuse of language, in the spirit of Kydland and Prescott (1980) and Chang (1998) we shall use u_{t+1} to describe what we shall dub a 'promised marginal value' that a competitive equilibrium offers to a representative firm.*

Remark 7.2.6 *We should instead, perhaps with more accuracy, define a promised marginal value as $\beta(A_0 - A_1 Q_{t+1}) - \beta\tau_{t+1} + \beta d u_{t+1}$, since this is the object to which the firm's first-order condition instructs it to equate to the marginal cost du_t of $u_t = q_{t+1} - q_t$.*[3] *But given (u_t, Q_t), the representative firm knows (Q_{t+1}, τ_{t+1}), so it is adequate to take u_{t+1} as the intermediate variable that summarizes how $\vec{\tau}_{t+1}$ affects the firm's choice of u_t.*

Definition 7.2.7 *Define a history $Q^t = [Q_0, \ldots, Q_t]$. A history-dependent tax policy is a sequence of functions $\{\sigma_t\}_{t=0}^{\infty}$ with time t component σ_t mapping Q^t into a choice of τ_{t+1}.*

Below we shall study history-dependent tax policies that either (a) solve a Ramsey plan or (b) are credible. We shall describe recursive representations of both types of history-dependent policies.

7.3 Ramsey problem

The planner's objective is cast in terms of consumer surplus net of the firm's adjustment costs. Consumer surplus is

$$\int_0^Q [A_0 - A_1 x]dx = A_0 Q - \frac{A_1}{2}Q^2,$$

so the planner's one-period return function is

$$A_0 Q_t - \frac{A_1}{2}Q_t^2 - \frac{d}{2}u_t^2. \tag{7.6}$$

At time 0, a Ramsey planner faces the intertemporal budget constraint

$$\sum_{t=1}^{\infty} \beta^t \tau_t Q_t = G_0. \tag{7.7}$$

Note that (7.7) precludes taxation of initial output Q_0.

[3] This choice would align better with how Chang (1998) chose to express his competitive equilibrium recursively.

Definition 7.3.1 *The Ramsey problem is to choose a tax sequence $\vec{\tau}$ and a competitive equilibrium outcome $(\vec{Q}, \vec{u})$ that maximize*

$$\sum_{t=0}^{\infty} \beta^t \left[A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 \right] \tag{7.8}$$

subject to (7.7).

Definition 7.3.2 *Ramsey timing protocol.*

1. *At time 0, knowing (Q_0, G_0), the Ramsey planner chooses $\{\tau_{t+1}\}_{t=0}^{\infty}$.*
2. *Given $(Q_0, \{\tau_{t+1}\}_{t=0}^{\infty})$, a competitive equilibrium outcome $\{u_t, Q_{t+1}\}_{t=0}^{\infty}$ emerges (see Definition 7.2.1).*

Remark 7.3.3 *In bringing out the timing protocol associated with a Ramsey plan, we run head on to a set of issues analysed by Bassetto (2005). This is because in Definition 7.3.2 of the Ramsey timing protocol, we have not completely described conceivable actions by the government and firms as time unfolds. For example, we are silent about how the government would respond if firms, for some unspecified reason, were to choose to deviate from the competitive equilibrium associated with the Ramsey plan, thereby possibly violating budget balance (7.7). Our definition of a Ramsey plan says nothing about how the government would respond. This is an example of the issues raised by Bassetto (2005), who identifies a class of government policy problems whose proper formulation requires supplying a complete and coherent description of all actors' behaviour across all possible histories. Implicitly, we are assuming that a more complete description of a government strategy than we have included could be specified that (a) agrees with ours along the Ramsey outcome, and (b) suffices uniquely to implement the Ramsey plan by deterring firms taking actions that deviate from the Ramsey outcome path.*

7.3.1 *Computing a Ramsey plan*

The planner chooses $\{u_t\}_{t=0}^{\infty}, \{\tau_t\}_{t=1}^{\infty}$ to maximize (7.8) subject to (7.4), (7.5), and (7.7). To formulate this problem as a Lagrangian, attach a Lagrange multiplier μ to the budget constraint (7.7). Then the planner chooses $\{u_t\}_{t=0}^{\infty}, \{\tau_t\}_{t=1}^{\infty}$ to maximize and the Lagrange multiplier μ to minimize

$$\sum_{t=0}^{\infty} \beta^t \left[A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 \right] + \mu \left[\sum_{t=0}^{\infty} \beta^t \tau_t Q_t - G_0 - \tau_0 Q_0 \right] \tag{7.9}$$

subject to (7.4) and (7.5).

7.4 Implementability multiplier approach

The Ramsey problem is a special case of the linear quadratic dynamic Stackelberg problem analysed in Ljungqvist and Sargent (2004, Ch. 18). The idea is to construct a recursive representation of a Ramsey plan by taking as state variables Lagrange multipliers on implementability constraints that require the Ramsey planner to choose among competitive equilibrium allocations. The motion through time of these Lagrange multipliers become components of a recursive representation of a history-dependent plan for taxes. For us, the key implementability conditions are (7.4) for $t \geq 0$.

Holding fixed μ and G_0, the Lagrangian (7.9) for the planning problem can be abbreviated as

$$\max_{\{u_t\},\{\tau_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 + \mu \tau_t Q_t \right].$$

Define

$$y_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix} = \begin{pmatrix} 1 \\ Q_t \\ \tau_t \\ u_t \end{pmatrix},$$

where $z_t = \begin{pmatrix} 1 \\ Q_t \\ \tau_t \end{pmatrix}$ are genuine state variables and u_t is a jump variable. We include τ_t as a state variable for bookkeeping purposes: it helps to map the problem into a linear regulator problem with no cross products between states and controls. However, it will be a redundant state variable in the sense that the optimal tax τ_{t+1} will not depend on τ_t. The government chooses τ_{t+1} at time t as a function of the time t state. Thus, we can rewrite the Ramsey problem as

$$\max_{\{y_t\},\{\tau_{t+1}\}} -\sum_{t=0}^{\infty} \beta^t y_t' R y_t \tag{7.10}$$

subject to z_0 given and the law of motion

$$\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = A \begin{pmatrix} z_t \\ u_t \end{pmatrix} + B \tau_{t+1}, \tag{7.11}$$

where

$$R = \begin{pmatrix} 0 & -\frac{A_0}{2} & 0 & 0 \\ -\frac{A_0}{2} & \frac{A_1}{2} & \frac{-\mu}{2} & 0 \\ 0 & \frac{-\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{2} \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -\frac{A_0}{d} & \frac{A_1}{d} & 0 & \frac{A_1}{d} + \frac{1}{\beta} \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{d} \end{pmatrix}.$$

Because this problem falls within the Ljungqvist and Sargent (2004, Ch. 18) framework, we can proceed as follows. Letting λ_t be a vector of Lagrangian multipliers on the transition laws summarized in Eq. (7.11), it follows that $\lambda_t = Py_t$, where P solves the Riccati equation

$$P = R + \beta A'PA - \beta^2 A'PB(\beta B'PB)^{-1}B'PA$$

and $\tau_{t+1} = -Fy_t$, where

$$F = \beta(\beta B'PB)^{-1}B'PA.$$

This we can rewrite as

$$\begin{pmatrix} \lambda_{zt} \\ \lambda_{ut} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix}.$$

Solve for u_t to get

$$u_t = -P_{22}^{-1}P_{21}z_t + P_{22}^{-1}\lambda_{ut},$$

where now the multiplier λ_{ut} becomes our authentic state variable, one that measures the costs of confirming the public's prior expectations about time t government actions. Then the complete state at time t becomes $\begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix}$. Thus,

$$y_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{pmatrix} \begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix}$$

so

$$\tau_{t+1} = -F \begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{pmatrix} \begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix}.$$

The evolution of the state is

$$\begin{pmatrix} z_{t+1} \\ \lambda_{ut+1} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ P_{21} & P_{22} \end{pmatrix} (A - BF) \begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{pmatrix}}_{G} \begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix}$$

with initial state

$$\begin{pmatrix} z_0 \\ \lambda_{u0} \end{pmatrix} = \begin{pmatrix} 1 \\ Q_0 \\ \tau_0 \\ 0 \end{pmatrix}. \tag{7.12}$$

Equation (7.12) incorporates the Ljungqvist and Sargent (2004, Ch. 18) finding that the Ramsey planner finds it optimal to set λ_{u0} to 0.

7.5 Kydland–Prescott (1980) approach

Kydland and Prescott (1980) or Chang (1998) would formulate our Ramsey problem in terms of the Bellman equation

$$v(Q_t, \tau_t, u_t) = \max_{\tau_{t+1}} \left\{ A_0 Q_t - \frac{A_1}{2}Q_t^2 - \frac{d}{2}u_t^2 + \mu\tau_t Q_t + \beta v(Q_{t+1}, \tau_{t+1}, u_{t+1}) \right\},$$

where the maximization is subject to the constraints

$$Q_{t+1} = Q_t + u_t$$

and

$$u_{t+1} = -\frac{A_0}{d} + \frac{A_1}{d}Q_t + \left(\frac{A_1}{d} + \frac{1}{\beta}\right)u_t + \frac{1}{d}\tau_{t+1}.$$

We now regard u_t as a state. It plays the role of a promised marginal utility in the Kydland and Prescott (1980) framework. Define the state vector to be

$$y_t = \begin{pmatrix} 1 \\ Q_t \\ \tau_t \\ u_t \end{pmatrix} = \begin{pmatrix} z_t \\ u_t \end{pmatrix},$$

where $z_t = \begin{pmatrix} 1 \\ Q_t \\ \tau_t \end{pmatrix}$ are authentic state variables and u_t is a variable whose time 0 value is a 'jump' variable but whose values for dates $t \geq 1$ will become state variables that encode history dependence in the Ramsey plan. Write a dynamic programming problem in the style of Kydland and Prescott (1980) as

$$v(y_t) = \max_{\tau_{t+1}} \left\{ -y_t'Ry_t + \beta v(y_{t+1}) \right\}, \tag{7.13}$$

where the maximization is subject to the constraint

$$y_{t+1} = Ay_t + B\tau_{t+1},$$

where

$$R = \begin{pmatrix} 0 & -\frac{A_0}{2} & 0 & 0 \\ -\frac{A_0}{2} & \frac{A_1}{2} & \frac{-\mu}{2} & 0 \\ 0 & \frac{-\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{2} \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -\frac{A_0}{d} & \frac{A_1}{d} & 0 & \frac{A_1}{d} + \frac{1}{\beta} \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{d} \end{pmatrix}.$$

Functional equation (7.13) is an optimal linear regulator problem. It has solution

$$v(y_t) = -y_t' P y_t,$$

where P solves

$$P = R + A'PA - A'PB(B'PB)^{-1}B'PA$$

and the optimal policy function is given by

$$\tau_{t+1} = -Fy_t, \tag{7.14}$$

where

$$F = \beta(\beta B'PB)^{-1}B'PA = (B'PB)^{-1}B'PA. \tag{7.15}$$

Note that since the formulas for A, B, and R are identical it follows that F and P are the same as in the Lagrangian multiplier approach of Section 7.4. The optimal choice of u_0 satisfies

$$\frac{\partial v}{\partial u_0} = 0.$$

If we partition P as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

then we have

$$0 = \frac{\partial}{\partial u_0}\left(z_0' P_{11} z_0 + z_0' P_{12} u_0 + u_0' P_{21} z_0 + u_0' P_{22} u_0\right)$$
$$= P_{12}' z_0 + P_{21} u_0 + 2P_{22} u_0,$$

which implies that

$$u_0 = -P_{22}^{-1} P_{21} z_0. \tag{7.16}$$

Thus, the Ramsey plan is

$$\tau_{t+1} = -F\begin{pmatrix} z_t \\ u_t \end{pmatrix} \text{ and } \begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = (A - BF)\begin{pmatrix} z_t \\ u_t \end{pmatrix},$$

with initial state $\begin{pmatrix} z_0 \\ -P_{22}^{-1} P_{21} z_0 \end{pmatrix}$.

7.5.1 *Comparison of the two approaches*

We can compare the outcome from the Kydland–Prescott approach to the outcome of the Lagrangian multiplier on the implementability constraint approach of Section 7.4. Using the formula

$$\begin{pmatrix} z_t \\ u_t \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{pmatrix}\begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix}$$

and applying it to the evolution of the state

$$\begin{pmatrix} z_{t+1} \\ \lambda_{ut+1} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ P_{21} & P_{22} \end{pmatrix}(A - BF)\begin{pmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{pmatrix}}_{G}\begin{pmatrix} z_t \\ \lambda_{ut} \end{pmatrix},$$

we get

$$\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = (A - BF)\begin{pmatrix} z_t \\ u_t \end{pmatrix} \tag{7.17}$$

or

$$y_{t+1} = A_F y_t, \tag{7.18}$$

where $A_F \equiv A - BF$. Then using the initial state value $\lambda_{u,0} = 0$, we obtain

$$\begin{pmatrix} z_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} z_0 \\ -P_{22}^{-1} P_{21} z_0 \end{pmatrix}. \tag{7.19}$$

This is identical to the initial state delivered by the Kydland–Prescott approach. Therefore, as expected, the two approaches provide identical Ramsey plans.

7.6 Recursive representation

An outcome of the preceding results is that the Ramsey plan can be represented recursively as the choice of an initial marginal utility (or rate of growth of output) according to a function

$$u_0 = v(Q_0|\mu) \tag{7.20}$$

that obeys (7.19) and the following updating equations for $t \geq 0$:

$$\tau_{t+1} = \tau(Q_t, u_t|\mu) \tag{7.21}$$

$$Q_{t+1} = Q_t + u_t \tag{7.22}$$

$$u_{t+1} = u(Q_t, u_t|\mu). \tag{7.23}$$

We have conditioned the functions v, τ, and u by μ to emphasize how the dependence of F on G appears indirectly through the Lagrange multiplier μ. We'll discuss how to compute μ in Section 7.7, but first want to consider the following numerical example.

7.6.1 *Example*

We computed the Ramsey plan for the following parameter values: $[A_0, A_1, d, \beta, Q_0] = [100, 0.05, 0.2, 0.95, 100]$. Figure 7.1[4] reports the Ramsey plan for τ and the Ramsey outcome for Q, u for $t = 0, \ldots, 20$. The optimal decision rule is[5]

$$\tau_{t+1} = -248.0624 - 0.1242Q_t - 0.3347u_t. \tag{7.24}$$

Note how the Ramsey plan calls for a high tax at $t = 1$ followed by a perpetual stream of lower taxes. Taxing heavily at first, less later sets up a time-inconsistency problem that we'll characterize formally after first discussing how to compute μ.

[4] The computations are executed in Matlab programs Evans_Sargent_Main.m and ComputeG.m. ComputeG.m solves the Ramsey problem for a given μ and returns the associated tax revenues (see Section 7.7) and the matrices F and P. Evans_Sargent_Main.m is the main driving file and with ComputeG.m computes the time series plotted in Figure 7.1.

[5] As promised, τ_t does not appear in the Ramsey planner's decision rule for τ_{t+1}.

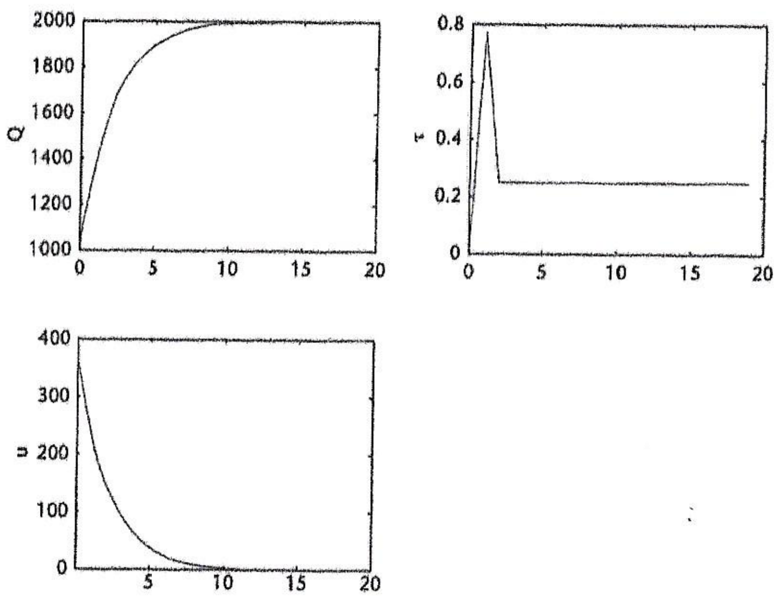

Figure 7.1 Ramsey plan and Ramsey outcome. From upper left to right: first panel, Q_t; second panel, τ_t; third panel, $u_t = Q_{t+1} - Q_t$.

7.7 Computing μ

Define the selector vectors $e_\tau = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}'$ and $e_Q = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}'$. Then express $\tau_t = e_\tau' y_t$ and $Q_t = e_Q' y_t$. Evidently, tax revenues $Q_t \tau_t = y_t' e_Q e_\tau' y_t = y_t' S y_t$ where $S \equiv e_Q e_\tau'$. We want to compute

$$T_0 = \sum_{t=1}^{\infty} \beta^t \tau_t Q_t = \beta \tau_1 Q_1 + \beta T_1,$$

where $T_1 = \sum_{t=2}^{\infty} \beta^{t-1} Q_t \tau_t$. The present values T_0 and T_1 are connected by

$$T_0 = \beta y_0' A_F' S A_F y_0 + \beta T_1.$$

Guess a solution that takes the form $T_t = y_t' \Omega y_t$ then find an Ω that satisfies

$$\Omega = \beta A_F' S A_F + \beta A_F' \Omega A_F. \tag{7.25}$$

Equation (7.25) is a discrete Lyapunov equation that can be solved for Ω using the Matlab program `dlyap` or `doublej2`.

The matrix F and therefore the matrix $A_F = A - BF$ depend on μ. To find a μ that guarantees that

$$T_0 = G, \tag{7.26}$$

we proceed as follows:

1. Guess an initial μ, compute a tentative Ramsey plan and the implied $T_0 = y_0'\Omega(\mu)y_0$.
2. If $T_0 > G$, lower μ; if $T_0 < \mu$, raise μ.
3. Continue iterating on step 3 until $T_0 = G$.

7.8 Time Inconsistency

Recall that the Ramsey planner chooses $\{u_t\}_{t=0}^{\infty}, \{\tau_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left[A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2\right]$$

subject to (7.4), (7.5), and (7.7). In this section, we note that a Ramsey plan is time-inconsistent, which we express as follows:

Proposition 7.8.1 *A continuation of a Ramsey plan is not a Ramsey plan.*

Let

$$w(Q_0, u_0|\mu_0) = \sum_{t=0}^{\infty} \beta^t \left[A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2\right], \tag{7.27}$$

where $\{Q_t, u_t\}_{t=0}^{\infty}$ are evaluated under the Ramsey plan whose recursive representation is given by (7.21), (7.22), (7.23) and where μ_0 is the value of the Lagrange multiplier that assures budget balance, computed as described in Section 7.7. Evidently, these continuation values satisfy the recursion

$$w(Q_t, u_t|\mu_0) = A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 + \beta w(Q_{t+1}, u_{t+1}|\mu_0) \tag{7.28}$$

for all $t \geq 0$, where $Q_{t+1} = Q_t + u_t$. Under the timing protocol affiliated with the Ramsey plan, the planner is committed to the outcome of iterations on (7.21), (7.22), (7.23). In particular, when time t comes, he

is committed to the value of u_t implied by the Ramsey plan and receives continuation value $w(Q_t, u_t|\mu_0)$.

That the Ramsey plan is time-inconsistent can be seen by subjecting it to the following 'revolutionary' test. First, define continuation revenues G_t that the government raises along the original Ramsey outcome by

$$G_t = \beta^{-1}(G_0 - \sum_{s=1}^{t} \beta^s \tau_s Q_s), \tag{7.29}$$

where $\{\tau_t, Q_t\}_{t=0}^{\infty}$ is the original Ramsey outcome.[6] Then at time $t \geq 1$, take (Q_t, G_t) inherited from the original Ramsey plan as initial conditions, and invite a brand new Ramsey planner to compute a new Ramsey plan, solving for a new u_t, to be called $\check{u}_t$, and for a new μ, to be called $\check{\mu}_t$. The revised Lagrange multiplier $\check{\mu}_t$ is chosen so that, under the new Ramsey Plan, the government is able to raise enough continuation revenues G_t given by (7.29). Would this new Ramsey plan be a continuation of the original plan? The answer is no because along a Ramsey plan, for $t \geq 1$, in general it is true that

$$w\left(Q_t, v(Q_t|\check{\mu}_t)|\check{\mu}_t\right) > w\left(Q_t, u_t|\mu_0\right), \tag{7.30}$$

which expresses a continuation Ramsey planner's incentive to deviate from a time 0 Ramsey plan by resetting u_t according to (7.20) and adjusting the Lagrange multiplier on the continuation appropriately to account for tax revenues already collected.[7] Inequality (7.30) expresses the time-inconsistency of a Ramsey plan.

To bring out the time inconsistency of the Ramsey plan, in Figure 7.2 we compare the time t values of τ_{t+1} under the original Ramsey plan with the value $\check{\tau}_{t+1}$ associated with a new Ramsey plan begun at time t with initial conditions (Q_t, G_t) generated by following the *original* Ramsey plan, where $G_t = \beta^{-t}(G_0 - \sum_{s=1}^{t} \beta^s \tau_s Q_s)$. Associated with the new Ramsey plan at t is a value μ_t of the Lagrange multiplier on the continuation government budget constraint. In Figure 7.3, we compare the time t outcome for u_t under the original Ramsey plan with the time t value of this new Ramsey problem starting from (Q_t, G_t). To

[6] The continuation revenues G_t are the time t present value of revenues that must be raised to satisfy the original time 0 government intertemporal budget constraint, taking into account the revenues already raised from $s = 1, \ldots, t$ under the original Ramsey plan.

[7] For example, let the Ramsey plan yield time 1 revenues $Q_1\tau_1$. Then at time 1, a continuation Ramsey planner would want to raise continuation revenues, expressed in units of time 1 goods, of $\tilde{G}_1 \equiv \frac{G - \beta Q_1 \tau_1}{\beta}$. To finance the remainder revenues, the continuation Ramsey planner would find a continuation Lagrange multiplier μ by applying the three-step procedure from the previous section to revenue requirements $\tilde{G}_1$.

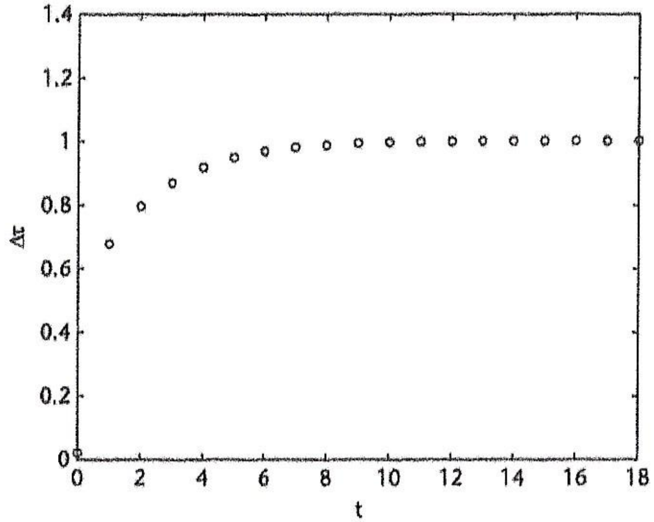

Figure 7.2 Difference $\check{\tau}_{t+1} - \tau_{t+1}$ where τ_{t+1} is along Ramsey plan and $\check{\tau}_{t+1}$ is for Ramsey plan restarted at t when Lagrange multiplier is frozen at μ_0.

compute u_t under the new Ramsey plan, we use the following version of formula (7.16):

$$\check{u}_t = -P_{22}^{-1}(\mu_t)P_{21}(\mu_t)z_t, \tag{7.31}$$

for z_t evaluated along the Ramsey outcome path, where we have included μ_t to emphasize the dependence of P on the Lagrange multiplier μ_0.[8] To compute u_t along the Ramsey path, we just iterate the recursion (7.17) starting from the initial Q_0 with u_0 being given by formula (7.16). Figure 7.2 plots the associated $\check{\tau}_{t+1} - \tau_{t+1}$. Figure 7.3, which plots $\check{u}_t - u_t$, indicates how far the reinitiated $\check{u}_t$ value departs from the time t outcome along the Ramsey plan. Note that the restarted plan raises the time $t+1$ tax and consequently lowers the time t value of u_t. Figure 7.4 plots the value of μ_t associated with the Ramsey plan that restarts at t with the required continued revenues G_t implied by the original Ramsey plan.

These figures help us understand the time inconsistency of the Ramsey plan. One feature to note is the large difference between $\check{\tau}_{t+1}$ and τ_{t+1} in Figure 7.2. If the government is able to reset to a new Ramsey plan at time t, it chooses a significantly higher tax rate than if it were

[8] It can be verified that this formula puts non-zero weight only on the components 1 and Q_t of z_t.

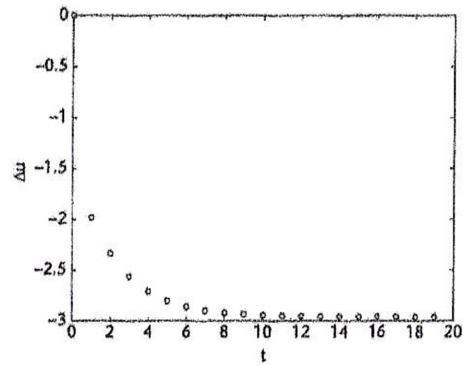

Figure 7.3 Difference $\check{u}_t - u_t$ where u_t is outcome along Ramsey plan and $\check{u}_t$ is for Ramsey plan restarted at t when Lagrange multiplier is frozen at μ_0.

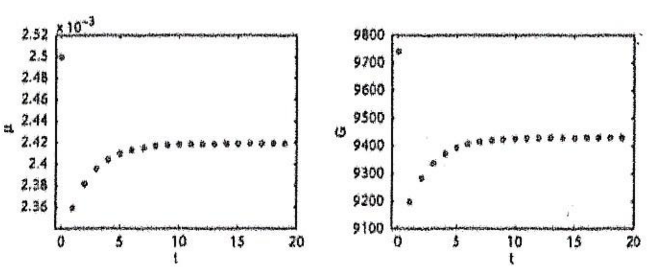

Figure 7.4 Value of Lagrange multiplier $\check{\mu}_t$ associated with Ramsey plan restarted at t (left), and the continuation G_t inherited from the original time 0 Ramsey plan G_t (right).

required to maintain the original Ramsey plan. The intuition here is that the government is required to finance a given present value of expenditures with distorting taxes τ. The quadratic adjustment costs prevent firms from reacting strongly to variations in the tax rate for next period, which tilts a time t Ramsey planner towards using time $t+1$ taxes. As was noted before, this is evident in Figure 7.1, where the government taxes the next period heavily and then falls back to a constant tax from then on. This can also been seen in Figure 7.4, where the government pays off a significant portion of the debt using the first period tax rate. The similarities between two graphs in Figure 7.4 reveals that there is a one-to-one mapping between G and μ. The Ramsey plan can then only be time consistent if G_t remains constant over time, which will not be true in general.

7.9 Credible policy

The theme of this section is conveyed in the following:

Remark 7.9.1 *We have seen that in general, a continuation of a Ramsey plan is not a Ramsey plan. This is sometimes summarized by saying that a Ramsey plan is not* credible. *A continuation of a credible plan is a credible plan.*

The literature on a credible public policy or credible plan introduced by Chari and Kehoe (1990) and Stokey (1989) describes history-dependent policies that arrange incentives so that public policies can be implemented by a *sequence* of government decision-makers. In this section, we sketch how recursive methods that Chang (1998) used to characterize credible policies would apply to our model.

A credibility problem arises because we assume that the timing of decisions differs from the Definition 7.3.1 Ramsey timing. Throughout this section, we now assume the following:

Definition 7.9.2 *Sequential timing protocol:*

1. *At each $t \geq 0$, given Q_t and expectations about a continuation tax policy $\{\tau_{s+1}\}_{s=t}^{\infty}$ and a continuation price sequence $\{p_{s+1}\}_{s=t}^{\infty}$, the representative firm chooses u_t.*
2. *At each t, given (Q_t, u_t), a government chooses τ_{t+1}.*

Item (2) captures that taxes are now set sequentially, the time $t+1$ tax being set after *the government has observed u_t.*

Of course, the representative firm sets u_t in light of its expectations of how the government will ultimately choose to set future taxes. A credible tax plan $\{\tau_{s+1}\}_{s=t}^{\infty}$ is one that is anticipated by the representative firm and also one that the government chooses to confirm.

We use the following recursion, closely related to but different from (7.28), to define the continuation value function for Ramsey planner:

$$J_t = A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 + \beta J_{t+1}(\tau_{t+1}, G_{t+1}). \tag{7.32}$$

This differs from (7.28) because continuation values are now allowed to depend explicitly on values of the choice τ_{t+1} and continuation government revenue to be raised G_{t+1} that need not be ones called for by the prevailing government policy. Thus, deviations from that policy are allowed, an alteration that recognizes that τ_t is chosen sequentially.

Express the government budget constraint as requiring that the $G = G_0$, where G_0 solves the difference equation

$$G_t = \beta \tau_{t+1} Q_{t+1} + \beta G_{t+1}, \tag{7.33}$$

subject to the terminal condition $\lim_{t \to +\infty} \beta^t G_t = 0$. Because the government is choosing sequentially, it is convenient to take G_t as a state variable at t and to regard the time t government as choosing τ_{t+1}, G_{t+1} subject to constraint (7.33).

To express the notion of a credible government plan concisely, we expand the strategy space by also adding J_t itself as a state variable and allow policies to take the following recursive forms.[9] Regard J_0 as a discounted present value promised to the Ramsey planner and take it as an initial condition. Then after choosing u_0 according to

$$u_0 = \upsilon(Q_0, G_0, J_0), \tag{7.34}$$

choose subsequent taxes, outputs, *and* continuation values according to recursions that can be represented as

$$\hat{\tau}_{t+1} = \tau(Q_t, u_t, G_t, J_t) \tag{7.35}$$

$$u_{t+1} = \xi(Q_t, u_t, G_t, J_t, \tau_{t+1}) \tag{7.36}$$

$$G_{t+1} = \beta^{-1} G_t - \tau_{t+1} Q_{t+1} \tag{7.37}$$

$$J_{t+1}(\tau_{t+1}, G_{t+1}) = \nu(Q_t, u_t, G_{t+1}, J_t, \tau_{t+1}). \tag{7.38}$$

Here $\hat{\tau}_{t+1}$ is the time $t+1$ government action called for by the plan, while τ_{t+1} is possibly some one-time deviation that the time $t+1$ government contemplates and G_{t+1} is the associated continuation tax collections. The plan is said to be *credible* if, for each t and each state (Q_t, u_t, G_t, J_t), the plan satisfies the incentive constraint

$$\begin{aligned} J_t &= A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 + \beta J_{t+1}(\hat{\tau}_{t+1}, \hat{G}_{t+1}) \\ &\geq A_0 Q_t - \frac{A_1}{2} Q_t^2 - \frac{d}{2} u_t^2 + \beta J_{t+1}(\tau_{t+1}, G_{t+1}) \end{aligned} \tag{7.39}$$

for all tax rates $\tau_{t+1} \in \mathbb{R}$ available to the government. Here $\hat{G}_{t+1} = \beta^{-1} G_t - \hat{\tau}_{t+1} Q_{t+1}$. Inequality (7.39) expresses that continuation values adjust to deviations in ways that discourage the government from deviating from the prescribed $\hat{\tau}_{t+1}$.

[9] This choice is the key to what Ljungqvist and Sargent (2004) call 'dynamic programming squared'.

Inequality (7.39) indicates that *two* continuation values J_{t+1} contribute to sustaining time t promised value J_t; $J_{t+1}(\hat{\tau}_{t+1}, \hat{G}_{t+1})$ is the continuation value when the government chooses to confirm e private sector's expectations, formed according to the decision rule (7.35);[10] $J_{t+1}(\tau_{t+1}, G_{t+1})$ tells the continuation consequences should the government disappoint the private sector's expectations. The internal structure of the plan deters deviations from it. That (7.39) maps *two* continuation values $J_{t+1}(\tau_{t+1}, G_{t+1})$ and $J_{t+1}(\hat{\tau}_{t+1}, \hat{G}_{t+1})$ into one promised value J_t reflects how a credible plan arranges a system of private sector expectations that induces the government to choose to confirm them. Chang (1998) builds on how inequality (7.39) maps two continuation values into one.

Remark 7.9.3 *Let* J *be the set of values associated with credible plans. Every value* $J \in$ J *can be attained by a credible plan that has a recursive representation of form* (7.35), (7.36), (7.37). *The set of values can be computed as the largest fixed point of an operator that maps sets of candidate values into sets of values. Given a value within this set, it is possible to construct a government strategy of the recursive form* (7.35), (7.36), (7.37) *that attains that value. In many cases, there is a set of values and associated credible plans. In those cases where the Ramsey outcome is credible, a multiplicity of credible plans must be a key part of the story because, as we have seen earlier, a continuation of a Ramsey plan is not a Ramsey plan. For it to be credible, a Ramsey outcome must be supported by a worse outcome associated with another plan, the prospect of reversion to which sustains the Ramsey outcome.*

7.10 Concluding remarks

The term 'optimal policy', which pervades an important applied monetary economics literature, means different things under different timing protocols. Under the 'static' Ramsey timing protocol (i.e., choose a sequence once-and-for-all), we obtain a unique plan. Here the phrase 'optimal policy' seems to fit well, since the Ramsey planner optimally reaps early benefits from influencing the private sector's beliefs about the government's later actions. But if we adopt the sequential timing protocol associated with credible public policies, 'optimal policy' is a more ambiguous description. There is a multiplicity of credible plans. True, the theory explains how it is optimal for the government to confirm the private sector's expectations about its actions along a

[10] Note the double role played by (7.35): as the decision rule for the government and as the private sector's rule for forecasting government actions.

credible plan but some credible plans have very bad outcomes. And these bad outcomes are central to the theory because it is the presence of bad credible plans that makes possible better ones by sustaining the low continuation values that appear in the second line of incentive constraint (7.39).

Recently, many have taken for granted that 'optimal policy' means 'follow the Ramsey plan'.[11] In pursuit of more attractive ways of describing a Ramsey plan when policy-making is in practice done sequentially, some writers have repackaged a Ramsey plan in the following way. Take a Ramsey *outcome*—a sequence of endogenous variables under a Ramsey plan—and reinterpret it (or perhaps only a subset of its variables) as a *target path* of relationships among outcome variables to be assigned to a sequence of policy-makers.[12] If appropriate (infinite dimensional) invertibility conditions are satisfied, it can happen that following the Ramsey plan is the *only* way to hit the target path.[13] The spirit of this work is to say, 'in a democracy we are obliged to live with the sequential timing protocol, so let's constrain policy makers' objectives in way that will force them to follow a Ramsey plan in spite of their benevolence'.[14] By this slight of hand, we acquire a theory of an optimal outcome target path.

This 'invertibility' argument leaves open two important loose ends: (1) implementation, and (2) time consistency. As for (1), repackaging a Ramsey plan (or the tail of a Ramsey plan) as a target outcome sequence does not confront the delicate issue of *how* that target path is to be implemented.[15] As for (2), it is an interesting question whether the 'invertibility' logic can repackage and conceal a Ramsey plan well enough to make policy-makers forget or ignore the benevolent intentions that give rise to the time inconsistency of a Ramsey plan in the first place. To attain such an optimal output path, policy-makers must forget their benevolent intentions because there will inevitably occur temptations to deviate from that target path, and the implied relationship among variables like inflation, output, and interest rates along it. The continuation of such an optimal target path is not an optimal target path.

[11] It is possible to read Woodford (2003) and Giannoni and Woodford (2010) as making some carefully qualified statements of this type. Some of the qualifications can be interpreted as advice 'eventually' to follow a tail of Ramsey plan.

[12] In our model, the Ramsey outcome would be a path $(\vec{p}, \vec{Q})$.

[13] See Giannoni and Woodford (2010).

[14] Sometimes the analysis is framed in terms of following the Ramsey plan only from some future date T onwards.

[15] See Bassetto (2005) and Atkeson et al. (2010).