7 THE CAPITAL ASSET PRICING MODEL

Two main problem types dominate the discipline of investment science. The first is to determine the best course of action in an investment situation. Problems of this type include how to devise the best portfolio, how to devise the optimal strategy for managing an investment, how to select from a group of potential investment projects, and so forth. Several examples of such problems were treated in Part 1 of this book. The second type of problem is to determine the correct, arbitrage-free, fair, or equilibrium price of an asset. We saw examples of this in Part 1 as well, such as the formula for the correct price of a bond in terms of the term structure of interest rates, and the formula for the appropriate value of a firm.

This chapter concentrates mainly on the pricing issue. It deduces the correct price of a risky asset within the framework of the mean–variance setting. The result is the **capital asset pricing model** (CAPM) developed primarily by Sharpe, Lintner, and Mossin, which follows logically from the Markowitz mean–variance portfolio theory described in the previous chapter. Later in this chapter we discuss how this result can be applied to investment decision problems.

7.1 MARKET EQUILIBRIUM

Suppose that everyone is a mean–variance optimizer as described in the previous chapter. Suppose further that everyone agrees on the probabilistic structure of assets; that is, everyone assigns to the returns of assets the same mean values, the same variances, and the same covariances. Furthermore, assume that there is a unique risk-free rate of borrowing and lending that is available to all, and that there are no transactions costs. With these assumptions what will happen?

From the one-fund theorem we know that everyone will purchase a single fund of risky assets, and they may, in addition, borrow or lend at the risk-free rate. Furthermore, since everyone uses the same means, variances, and covariances, everyone will use the same risky fund. The mix of these two assets, the risky fund and the risk-free asset, will likely vary across individuals according to their individual tastes for risk. Some will seek to avoid risk and will, accordingly, have a high percentage of the risk-free asset in their portfolios; others, who are more aggressive, will have a high percentage of the risky fund. However, every individual will form a portfolio that is a mix of the risk-free asset and the single, risky *one fund*. Hence the *one fund* in the theorem is really the *only fund* that is used.

If everyone purchases the same fund of risky assets, what must that fund be? The answer to this question is the key insight underlying the CAPM. A bit of reflection reveals that the answer is that this fund must equal the **market portfolio.** The market portfolio is the summation of all assets. In the world of equity securities, it is the totality of shares of IBM, GM, DIS, and so forth. If everyone buys just one fund, and their purchases add up to the market, then that one fund must be the market as well; that is, it must contain shares of every stock in proportion to that stock's representation in the entire market.

An asset's weight in a portfolio is defined as the proportion of portfolio capital that is allocated to that asset. Hence the weight of an asset in the market portfolio is equal to the proportion of that asset's total capital value to the total market capital value. These weights are termed **capitalization weights.** It is these weights that we usually denote by w_i. In other words, the w_i's of the market portfolio are the capitalization weights of the assets.

The exact definition of the market portfolio is illustrated as follows. Suppose there are only three stocks in the market: Jazz, Inc., Classical, Inc., and Rock, Inc. Their outstanding shares and prices are shown in Table 7.1. The market weights are proportional to the total market capitalization, not to the number of shares.

In the situation where everyone follows the mean–variance methodology with the same estimates of parameters, we know that the efficient fund of risky assets will be the market portfolio. Hence under these assumptions there is no need for us to formulate the mean–variance problem, to estimate the underlying parameters, or to solve the system of equations that define the optimal portfolio. We know that the optimal portfolio will turn out to be the market portfolio.

TABLE 7.1
Market Capitalization Weights

Security	Shares outstanding	Relative shares in market	Price	Capitalization	Weight in market
Jazz, Inc.	10,000	1/8	\$6.00	\$60,000	3/20
Classical, Inc.	30,000	3/8	\$4.00	\$120,000	3/10
Rock, Inc.	40,000	1/2	\$5.50	\$220,000	11/20
Total	80,000	1		\$400,000	1

The percentage of shares of a stock in the market portfolio is a share-weighted proportion of total shares. These percentages are not *the market portfolio weights. The market portfolio weight of a stock is proportional to capitalization. If the price of an asset changes, the share proportions do not change, but the capitalization weights do change.*

How does this happen? How can it be that we solve the problem even without knowing the required data? The answer is based on an **equilibrium** argument. If everyone else (or at least a large number of people) solves the problem, we do not need to. It works like this: The return on an asset depends on both its initial price and its final price. The other investors solve the mean–variance portfolio problem using their common estimates, and they place orders in the market to acquire their portfolios. If the orders placed do not match what is available, the prices must change. The prices of assets under heavy demand will increase; the prices of assets under light demand will decrease. These price changes affect the estimates of asset returns directly, and hence investors will recalculate their optimal portfolios. This process continues until demand exactly matches supply; that is, it continues until there is equilibrium.

In the idealized world, where every investor is a mean–variance investor and all have the same estimates, everyone buys the same portfolio, and that must be equal to the market portfolio. In other words, prices adjust to drive the market to efficiency. Then after other people have made the adjustments, we can be sure that the efficient portfolio is the market portfolio, so we need not make any calculations.

This theory of equilibrium is usually applied to assets that are traded repeatedly over time, such as the stock market. In this case it is argued that individuals adjust their return estimates slowly, and only make a series of minor adjustments to their calculations rather than solving the entire portfolio optimization problem at one time.

Finally, in such equilibrium models it is argued that the appropriate equilibrium need be calculated by only a few devoted (and energetic) individuals. They move prices around to the proper value, and other investors follow their lead by purchasing the market portfolio.

These arguments about the equilibrium process all have a degree of plausibility, and all have weaknesses. Deeper analysis can be carried out, but for our purposes we will merely consider that equilibrium occurs. Hence the ultimate conclusion of the mean–variance approach is that the *one fund* must be the market portfolio.

7.2 THE CAPITAL MARKET LINE

Given the preceding conclusion that the single efficient fund of risky assets is the market portfolio, we can label this fund on the $\bar{r}$–σ diagram with an M for *market*. The efficient set therefore consists of a single straight line, emanating from the risk-free point and passing through the market portfolio. This line, shown in Figure 7.1, is called the **capital market line.**

This line shows the relation between the expected rate of return and the risk of return (as measured by the standard deviation) for efficient assets or portfolios of assets. It is also referred to as a pricing line, since prices should adjust so that efficient assets fall on this line.

The line has great intuitive appeal. It states that as risk increases, the corresponding expected rate of return must also increase. Furthermore, this relationship can be

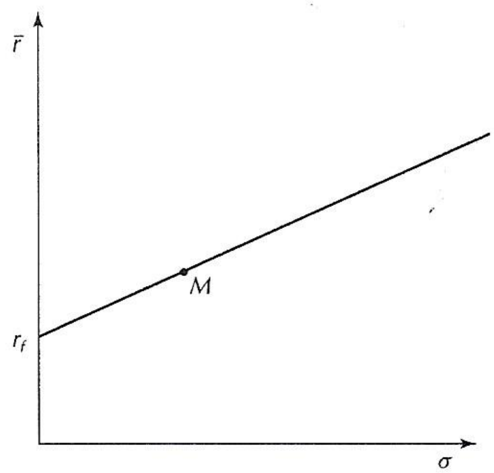

FIGURE 7.1 Capital market line. Efficient assets must all lie on the line determined by the risk-free rate and the market portfolio.

described by a straight line if risk is measured by standard deviation. In mathematical terms the capital market line states that

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma \tag{7.1}$$

where $\bar{r}_M$ and σ_M are the expected value and the standard deviation of the market rate of return and $\bar{r}$ and σ are the expected value and the standard deviation of the rate of return of an arbitrary efficient asset.

The slope of the capital market line is $K = (\bar{r}_M - r_f)/\sigma_M$, and this value is frequently called the **price of risk.** It tells by how much the expected rate of return of a portfolio must increase if the standard deviation of that rate increases by one unit.

Example 7.1 (The impatient investor) Mr. Smith is young and impatient. He notes that the risk-free rate is only 6% and the market portfolio of risky assets has an expected return of 12% and a standard deviation of 15%. He figures that it would take about 60 years for his $1,000.00 nest egg to increase to $1 million if it earned the market rate of return. He can't wait that long. He wants that $1 million in 10 years.

Mr. Smith easily determines that he must attain an average rate of return of about 100% per year to achieve his goal (since $\$1{,}000 \times 2^{10} = \$1{,}048{,}000$). Correspondingly, his yearly standard deviation according to the capital market line would be the value of σ satisfying

$$1.0 = .06 + \frac{.12 - .06}{.15}\sigma$$

or $\sigma = 10$. This corresponds to $\sigma = 1{,}000\%$. So this young man is certainly not guaranteed success (even if he could borrow the amount required to move far beyond the market on the capital market line).

Example 7.2 (An oil venture) Consider an oil drilling venture. The price of a share of this venture is \$875. It is expected to yield the equivalent of \$1,000 after 1 year, but due to high uncertainty about how much oil is at the drilling site, the standard deviation of the return is $\sigma = 40\%$. Currently the risk-free rate is 10%. The expected rate of return on the market portfolio is 17%, and the standard deviation of this rate is 12%.

Let us see how this venture compares with assets on the capital market line. Given the level of σ, the expected rate of return predicted by the capital market line is

$$\overline{r} = .10 + \frac{.17 - .10}{.12} .40 = 33\%.$$

However, the actual expected rate of return is only $\overline{r} = 1{,}000/875 - 1 = 14\%$. Therefore the point representing the oil venture lies well below the capital market line. (This does *not* mean that the venture is necessarily a poor one, as we shall see later, but it certainly does not, by itself, constitute an efficient portfolio.)

7.3 THE PRICING MODEL

The capital market line relates the expected rate of return of an efficient portfolio to its standard deviation, but it does not show how the expected rate of return of an individual asset relates to its individual risk. This relation is expressed by the capital asset pricing model.

We state this major result as a theorem. The reader may wish merely to glance over the proof at first reading since it is a bit involved. We shall discuss the implications of the result following the proof.

The capital asset pricing model (CAPM) *If the market portfolio M is efficient, the expected return $\overline{r}_i$ of any asset i satisfies*

$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f) \tag{7.2}$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}. \tag{7.3}$$

Proof: For any α consider the portfolio consisting of a portion α invested in asset i and a portion $1 - \alpha$ invested in the market portfolio M. (We allow $\alpha < 0$, which corresponds to borrowing at the risk-free rate.) The expected rate of return of this portfolio is

$$\overline{r}_\alpha = \alpha\overline{r}_i + (1 - \alpha)\overline{r}_M$$

and the standard deviation of the rate of return is

$$\sigma_\alpha = [\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2]^{1/2}.$$

As α varies, these values trace out a curve in the $\overline{r}$–σ diagram, as shown in Figure 7.2. In particular, $\alpha = 0$ corresponds to the market portfolio M. This

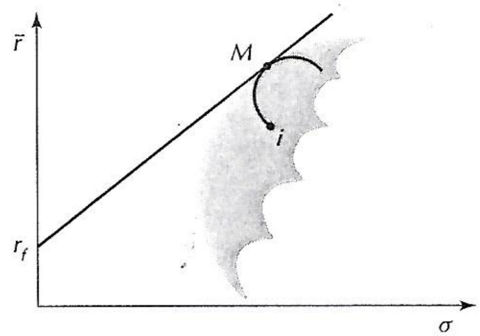

FIGURE 7.2 Portfolio curve. The family of portfolios traces out a curve on the diagram. This curve cannot cross the capital market line, and hence must be tangent to that line.

curve cannot cross the capital market line. If it did, the portfolio corresponding to a point above the capital market line would violate the very definition of the capital market line as being the efficient boundary of the feasible set. Hence as α passes through zero, the curve must be tangent to the capital market line at M. This tangency is the condition that we exploit to derive the formula.

The tangency condition can be translated into the condition that the slope of the curve is equal to the slope of the capital market line at the point M. To set up this condition we need to calculate a few derivatives.

First we have

$$\frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}.$$

Thus,

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}.$$

We then use the relation

$$\frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{d\bar{r}_\alpha/d\alpha}{d\sigma_\alpha/d\alpha}$$

to obtain

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}.$$

This slope must equal the slope of the capital market line. Hence,

$$\frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r_f}{\sigma_M}.$$

We now just solve for $\bar{r}_i$, obtaining the final result

$$\bar{r}_i = r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M^2}\right)\sigma_{iM} = r_f + \beta_i(\bar{r}_M - r_f).$$

This is clearly equivalent to the stated formula. ▮

The value β_i is referred to as the **beta** of an asset. When the asset is fixed in a discussion, we often just write beta without a subscript—β. An asset's beta is all that need be known about the asset's risk characteristics to use the CAPM formula.

The value $\overline{r}_i - r_f$ is termed the **expected excess rate of return** of asset i; it is the amount by which the rate of return is expected to exceed the risk-free rate. Likewise, $\overline{r}_M - r_f$ is the expected excess rate of return of the market portfolio. In terms of these expected excess rates of return, the CAPM says that the expected excess rate of return of an asset is proportional to the expected excess rate of return of the market portfolio, and the proportionality factor is β. So with r_f taken as a base point, the expected returns of a particular asset and of the market above that base are proportional.

An alternative interpretation of the CAPM formula is based on the fact that β is a normalized version of the covariance of the asset with the market portfolio. Hence the CAPM formula states that the expected excess rate of return of an asset is directly proportional to its covariance with the market. It is this covariance that determines the expected excess rate of return.

To gain insight into this result, let us consider some extreme cases. Suppose, first, that the asset is completely *uncorrelated* with the market; that is, $\beta = 0$. Then, according to the CAPM, we have $\overline{r} = r_f$. This is perhaps at first sight a surprising result. It states that even if the asset is very risky (with large σ), the expected rate of return will be that of the risk-free asset—there is no premium for risk. The reason for this is that the risk associated with an asset that is uncorrelated with the market can be diversified away. If we had many such assets, each uncorrelated with the others and with the market, we could purchase small amounts of each of them, and the resulting total variance would be small. Since the final composite return would have small variance, the corresponding expected rate of return should be close to r_f.

Even more extreme is an asset with a negative value of β. In that case $\overline{r} < r_f$; that is, even though the asset may have very high risk (as measured by its σ), its expected rate of return should be even less than the risk-free rate. The reason is that such an asset reduces the overall portfolio risk when it is combined with the market. Investors are therefore willing to accept the lower expected value for this risk-reducing potential. Such assets provide a form of insurance. They do well when everything else does poorly.

The CAPM changes our concept of the risk of an asset from that of σ to that of β. It is still true that, overall, we measure the risk of a portfolio in terms of σ, but this does not translate into a concern for the σ's of individual assets. For those, the proper measure is their β's.

Example 7.3 (A simple calculation) We illustrate how simple it is to use the CAPM formula to calculate an expected rate of return. Let the risk-free rate be $r_f = 8\%$. Suppose the rate of return of the market has an expected value of 12% and a standard deviation of 15%.

Now consider an asset that has covariance of .045 with the market. Then we find $\beta = .045/(.15)^2 = 2.0$. The expected return of the asset is $\overline{r} = .08 + 2 \times (.12 - .08) = .16 = 16\%$.

Betas of Common Stocks

The concept of beta is well established in the financial community, and it is referred to frequently in technical discussions about particular stocks. Beta values are estimated by various financial service organizations. Typically, these estimates are formed by using a record of past stock values (usually about 6 or 18 months of weekly values) and computing, from the data, average values of returns, products of returns, and squares of returns in order to approximate expected returns, covariances, and variances. The beta values so obtained drift around somewhat over time, but unless there are drastic changes in a company's situation, its beta tends to be relatively stable.

Table 7.2 lists some well-known U.S. companies and their corresponding beta (β) and volatility (σ) values as estimated at a particular date. Try scanning the list and see if the values given support your intuitive impression of the company's market

TABLE 7.2
Some U.S. Companies: Their Betas and Sigmas

Ticker sym	Company name	Beta	Volatility
KO	Coca-Cola Co	1.19	18%
DIS	Disney Productions	2.23	22%
EK	Eastman Kodak	1.43	34%
XON	Exxon Corp	.67	18%
GE	General Electric CO	1.26	15%
GM	General Motors Corp	.81	19%
GS	Gillette Co	1.09	21%
HWP	Hewlett-Packard Co	1.65	21%
HIA	Holiday Inns Inc	2.56	39%
KM	K-Mart Corp	.82	20%
LK	Lockheed Corp	3.02	43%
MCD	McDonalds Corp	1.56	21%
MRK	Merck & Co	.94	20%
MMM	Minnesota Mining & Mfg	1.00	17%
JCP	Penny J C Inc	1.22	20%
MO	Phillip Morris Inc.	.87	21%
PG	Procter & Gamble	.70	14%
SA	Safeway Stores Inc	.72	14%
S	Sears Roebuck & Co	1.04	19%
SD	Standard Oil of Calif	.85	24%
SYN	Syntex Corp	1.18	31%
TXN	Texas Instruments	1.46	23%
X	US Steel Corp	1.03	26%
UNP	Union Pacific Corp	.65	18%
ZE	Zenith Radio Corp	2.01	32%

Source: *Dailygraph Stock Option Guide*, William O'Neil & Co, Inc., Los Angeles, December 7, 1979. Reprinted with permission of Daily Graphs, P.O. Box 66919, Los Angeles, CA 90066.

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properties. Generally speaking, we expect aggressive companies or highly leveraged companies to have high betas, whereas conservative companies whose performance is unrelated to the general market behavior are expected to have low betas. Also, we expect that companies in the same business will have similar, but not identical, beta values. Compare, for instance, JC Penny with Sears Roebuck, or Exxon with Standard Oil of California.

Beta of a Portfolio

It is easy to calculate the overall beta of a portfolio in terms of the betas of the individual assets in the portfolio. Suppose, for example, that a portfolio contains n assets with the weights $w_1, w_2, \ldots, w_n$. The rate of return of the portfolio is $r = \sum_{i=1}^{n} w_i r_i$. Hence $\text{cov}(r, r_M) = \sum_{i=1}^{n} w_i \, \text{cov}(r_i, r_M)$. It follows immediately that

$$\beta_p = \sum_{i=1}^{n} w_i \beta_i \, . \tag{7.4}$$

In other words, the portfolio beta is just the weighted average of the betas of the individual assets in the portfolio, with the weights being identical to those that define the portfolio.

7.4 THE SECURITY MARKET LINE

The CAPM formula can be expressed in graphical form by regarding the formula as a linear relationship. This relationship is termed the **security market line.** Two versions are shown in Figure 7.3.

Both graphs show the linear variation of $\bar{r}$. The first expresses it in covariance form, with $\text{cov}(r, r_M)$ being the horizontal axis. The market portfolio corresponds to the point σ_M^2 on this axis. The second graph shows the relation in beta form, with beta being the horizontal axis. In this case the market corresponds to the point $\beta = 1$.

Both of these lines highlight the essence of the CAPM formula. Under the equilibrium conditions assumed by the CAPM, any asset should fall on the security market line.

The security market line expresses the risk–reward structure of assets according to the CAPM, and emphasizes that the risk of an asset is a function of its covariance with the market or, equivalently, a function of its beta.

Systematic Risk

The CAPM implies a special structural property for the return of an asset, and this property provides further insight as to why beta is the most important measure of risk. To develop this result we write the (random) rate of return of asset i as

$$r_i = r_f + \beta_i (r_M - r_f) + \varepsilon_i \, . \tag{7.5}$$

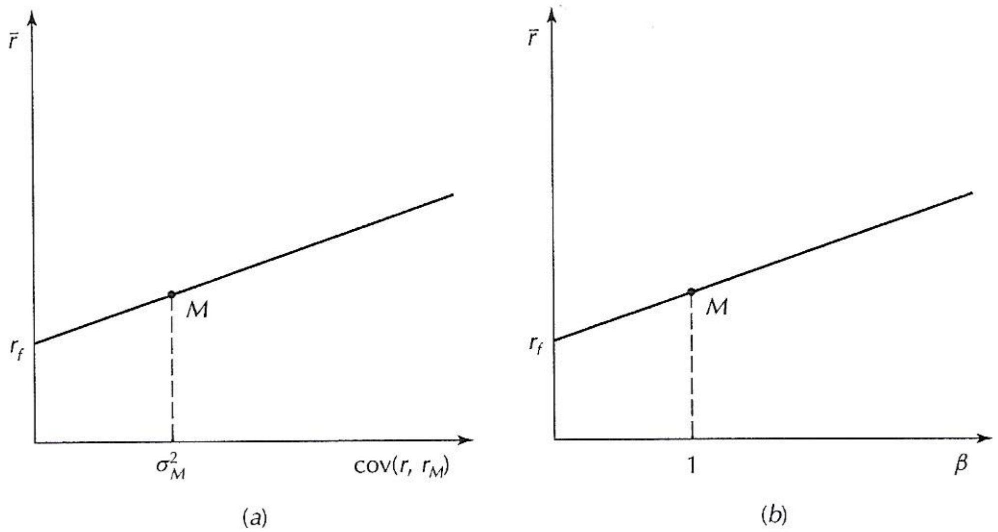

FIGURE 7.3 Security market line. The expected rate of return increases linearly as the covariance with the market increases or, equivalently, as β increases.

This is just an arbitrary equation at this point. The random variable ε_i is chosen to make it true. However, the CAPM formula tells us several things about ε_i.

First, taking the expected value of (7.5), the CAPM says that $E(\varepsilon_i) = 0$. Second, taking the correlation of (7.5) with r_M (and using the definition of β_i), we find $\text{cov}(\varepsilon_i, \sigma_M) = 0$. We can therefore write

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\varepsilon_i)$$

and we see that σ_i^2 is the sum of two parts. The first part, $\beta_i^2 \sigma_M^2$, is termed the **systematic risk.** This is the risk associated with the market as a whole. This risk cannot be reduced by diversification because every asset with nonzero beta contains this risk. The second part, $\text{var}(\varepsilon_i)$, is termed the **nonsystematic, idiosyncratic,** or **specific risk.** This risk is uncorrelated with the market and can be reduced by diversification. It is the systematic (or nondiversifiable) risk, measured by beta, that is most important, since it directly combines with the systematic risk of other assets.

Consider an asset on the capital market line[1] with a value of β. The standard deviation of this asset is $\beta\sigma_M$. It has only systematic risk; there is no nonsystematic risk. This asset has an expected rate of return equal to $\bar{r} = r_f + \beta(\bar{r}_M - r_f)$. Now consider a whole group of other assets, all with the same value of β. According to CAPM, these all have the same expected rate of return, equal to $\bar{r}$. However, if these assets carry nonsystematic risk, they will not fall on the capital market line. Indeed, as the nonsystematic risk increases, the points on the $\bar{r}$–σ plane representing these assets drift to the right, as shown in Figure 7.4. The horizontal distance of a point from the capital market line is therefore a measure of the nonsystematic risk.

[1] Of course, to be exactly on the line, the asset must be equivalent to a combination of the market portfolio and the risk-free asset.

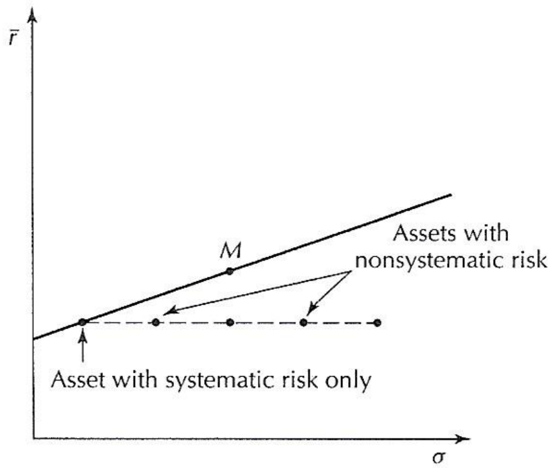

FIGURE 7.4 Systematic and nonsystematic risk. An asset on the capital market line has only systematic risk. Assets with nonsystematic risk fall to the right of the capital market line.

7.5 INVESTMENT IMPLICATIONS

The question of interest for the investor is: Can the CAPM help with investment decisions? There is not a simple answer to this question.

The CAPM states (or assumes), based on an equilibrium argument, that the solution to the Markowitz problem is that the market portfolio is the *one fund* (and *only fund*) of risky assets that anyone need hold. This fund is supplemented only by the risk-free asset. The investment recommendation that follows this argument is that an investor should simply purchase the market portfolio. That is, ideally, an investor should purchase a little bit of every asset that is available, with the proportions determined by the relative amounts that are issued in the market as a whole. If the world of equity securities is taken as the set of available assets, then each person should purchase some shares in every available stock, in proportion to the stocks' monetary share of the total of all stocks outstanding. It is not necessary to go to the trouble of analyzing individual issues and computing a Markowitz solution. Just buy the market portfolio.

Since it would be rather cumbersome for an individual to assemble the market portfolio, mutual funds have been designed to match the market portfolio closely. These funds are termed **index funds,** since they usually attempt to duplicate the portfolio of a major stock market index, such as the *Standard & Poor's 500* (S&P 500), an average of 500 stocks that as a group is thought to be representative of the market as a whole. Other indices use even larger numbers of stocks. A CAPM purist (that is, one who fully accepts the CAPM theory as applied to publicly traded securities) could just purchase one of these index funds (to serve as the *one fund*) as well as some risk-free securities such as U.S. Treasury bills.

Some people believe that they can do better than blindly purchasing the market portfolio. The CAPM, after all, assumes that everyone has identical information about the (uncertain) returns of all assets. Clearly, this is not the case. If someone believes that he or she possesses superior information, then presumably that person could form a portfolio that would outperform the market. We return to this issue in the next