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Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models*

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We study a class of models in which the law of motion perceived by agents influences the law of motion that they actually face. We assume that agents update their perceived law of motion by least squares. We show how the perceived law of motion and the actual one may converge to one another, depending on the behavior of a particular ordinary differential equation. The differential equation involves the operator that maps the perceived law of motion into the actual one. *Journal of Economic Literature* Classification Numbers: 021, 023, 211.

1. INTRODUCTION

This paper studies a class of linear stochastic models in which the actual law of motion depends on the law of motion that is perceived by the agents in the model. If the perceived law of motion at t is represented by β_t, then

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the actual law of motion at t is given by $T(\beta_t)$. We assume that over time, the agents in the model are learning about the law of motion. We assume a particular class of learning mechanisms (versions of least squares). We describe sets of restrictions on the learning mechanism and on the economic environment (operators T) that are sufficient to make β_t converge strongly to a rational expectations equilibrium (a fixed point of T). A sense will also be described in which the restrictions are necessary for strong convergence of β_t to a fixed point of T.

The models that we study are "adaptive" in the sense in which that term is used in the control literature.[1] In particular, the T mapping is derived under the assumption that agents behave as if they know with certainty that the true law of motion is time invariant and given by β_t. The models do not incorporate fully optimal behavior or rational expectations, because agents operate under the continually falsified assumption that the law of motion is time invariant and known for sure.[2] To facilitate analysis, we represent the learning algorithm recursively, in the form of a stochastic difference equation for β_t. We then describe a set of regularity conditions under which the limiting behavior of sample paths for β_t is governed by the limiting behavior of an associated ordinary differential equation, namely

$$(d/dt)\beta = T(\beta) - \beta. \tag{0}$$

We establish that the only possible limit points of the stochastic difference equation for β_t are the stationary points of (0), i.e., rational expectations equilibria. Further, whether β_t can converge almost surely to a fixed point of T is determined by the stability of (0) at that fixed point. These results extend, unify, and help us to interpret some earlier work on least squares learning (Mann and Wald [21]; Bray [4]; Bray and Savin [6]; Fourgeaud, Gourieroux, and Pradel [11]). We show how our methods apply to several models that had been studied with different methods by earlier researchers. We also illustrate how our method can be used to establish convergence for a model that is technically more intricate than those earlier models because it assumes that agents learn by regressing variables on lagged endogenous as well as exogenous variables.

We obtain our results by applying the "differential equation approach" to the study of almost sure convergence (Ljung [19] and Ljung and

[1] See Goodwin and Sin [13] and Ljung and Soderstrom [18] for surveys of adaptive filtering and control. They describe prediction rules and control laws that are "adaptive" in the sense that the rules (functions) change over time in a way designed to make them more consistent with the unknown laws of motion governing the environment.

[2] This point was stressed by Bray and Savin [6], and is equally applicable to their systems and ours.

Soderstrom [18]).[3] We proceed by imposing on our system enough conditions so that the convergence theorems of Ljung [19] can be applied.

The remainder of this paper is organized as follows. Section 2 describes the class of models under study and states three propositions about them. Section 3 briefly extends our results to comprehend simultaneous learning. Five examples of models of learning are studied in Section 4, while conclusions are given in Section 5. Appendix 1 states the theorems of Ljung [19] which we are using, while the remaining two appendixes contain proofs of some propositions stated in the text.

2. Convergence with Lagged Information

The position of the model at time t is described by an n-dimensional vector of random variables z_t. Let z_{1t} and z_{2t} be subvectors of z_t, not necessarily disjoint, of dimensions n_1 and n_2, respectively. Without loss of generality, we assume that z_t, z_{1t}, z_{2t} can be written as

$$z_t = \begin{bmatrix} z_{1t} \\ z_{1t}^c \end{bmatrix} = \begin{bmatrix} z_{2t}^c \\ z_{2t} \end{bmatrix}. \tag{1}$$

Here z_{1t} contains those variables whose future values agents care about, while z_{2t} contains those variables that agents use to predict z_{1t+1}. Each of z_1 and z_2 may contain exogenous and endogenous variables; z_1 and z_2 may have elements in common; and neither z_1 nor z_2 has to contain the other. It is always possible to partition z_t to satisfy the above equality: simply let the first elements of z_t be those which belong to z_1 and not to z_2, followed by elements that belong to z_1 and z_2, and then the elements that belong to z_2 and not to z_1.

We will study models with the following self-referential structure. Suppose that at time t agents believe that the law of motion for z_{1t} is given by

$$z_{1t} = \beta_t z_{2t-1} + \eta_t, \tag{2}$$

where η_t is orthogonal to all past z_2's, and $E\eta_t = 0$. That agents believe (2) causes the actual law of motion for the entire vector z_t to be given by

$$z_t = \begin{bmatrix} z_{1t} \\ z_{1t}^c \end{bmatrix} = \begin{bmatrix} 0 & T(\beta_t) \\ & A(\beta_t) \end{bmatrix} \begin{bmatrix} z_{2t-1}^c \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} V(\beta_t) \\ B(\beta_t) \end{bmatrix} u_t, \tag{3}$$

[3] Ljung and Soderstrom [18] describe how the differential equation approach generalizes the stochastic approximation method of Robbins and Monro [24]. Also see Kushner and Clark [16] for further developments of the Robbins–Monro method. Aoki [2] applied a Robbins–Monro algorithm to study convergence of a price adjustment scheme. Ljung and Soderstrom [18] apply Ljung's theorem [5] to study the convergence and asymptotic distribution of a variety of recursive least squares algorithms.

where u_t is an m-dimensional vector white noise satisfying $Eu_tu_t' = \Sigma$ for all t. The role of economic theory is to determine the mappings T, B, A, and V, whose dimensions are such that all matrix multiplications in (3) are well defined. Typically, z_{1t}^c will include only exogenous variables, in which case A and B are constant functions (being independent of β_t).[4] In the examples in Section 4, we illustrate how to obtain (3) from (2) for several particular economic models. Note that (3) makes z_t linear in z_{t-1} and u_t, but possibly nonlinear in β_t. The examples in Section 4 will illustrate how the "cross-equation restrictions" of rational expectations models can serve to make (3) nonlinear in β_t.

Note that (2) and (3) impose the restriction that on the right side of (2) there appear at least all of the variables that have nonzero coefficients in the true law of motion for z_1 given by (3). This restriction rules out models in which there are "hidden state variables" in the true law of motion for z_1, such as would be induced by various structures with private information. In Marcet and Sargent [22], we describe how the present setup could be modified to incorporate such hidden state variables.[5]

Agents' beliefs β_t evolve according to the following scheme. Let $\{\alpha_t\}$ be a positive, nondecreasing sequence of real numbers, with $\alpha_t \to 1$ as $t \to \infty$. Define $\bar{\beta}_t$ and $\bar{R}_t$ by

$$\begin{aligned} \bar{\beta}_t' &= \beta_{t-1}' + (\alpha_t/t)\, R_{t-1}^{-1}\{z_{2t-2}z_{2t-2}'[T(\beta_{t-1})' - \beta_{t-1}'] \\ &\quad + z_{2t-2}u_{t-1}'\, V(\beta_{t-1})'\} \\ \bar{R}_t &= R_{t-1} + (\alpha_t/t)[z_{2t-1}z_{2t-1}' - R_{t-1}/\alpha_t]. \end{aligned} \tag{4a}$$

Let $D_2 \subset D_1 \subset \mathbb{R}^{n_1 \times (n_2)^3}$. The algorithm for generating beliefs β_t is

$$(\beta_t, R_t) = \begin{cases} (\bar{\beta}_t, \bar{R}_t) & \text{if } (\bar{\beta}_t, \bar{R}_t) \in D_1 \\ \text{some value in } D_2 & \text{if } (\bar{\beta}_t, \bar{R}_t) \notin D_1. \end{cases} \tag{4b}$$

The most natural candidate for "some value in D_2" is $(\beta_{t'}, R_{t'})$ where t' is the last time t' that $(\beta_{t'}, R_{t'}) \in D_2$, but any other point in D_2 is acceptable.

[4] Throughout this paper, "exogenous variables" denotes elements of z_t whose evolution does not depend on the perceptions of the agents, β_t. By contrast, "endogenous variables" have an actual law of motion dependent on β_t.

[5] By introducing an element that is identically equal to one in z_{2t} and setting the corresponding row of B equal to zero, the present framework can cover the case in which agents do not know the means of z_1.

When $D_2 = D_1 = \mathbb{R}^{n_1 \times (n_2)^3}$, we have that $\bar{\beta}_t = \beta_t$ and $\bar{R}_t = R_t$ for all t. In this case, with appropriate initial conditions, (4) implies

$$\beta_t = \left[\sum_{i=1}^{t-1} \alpha_i z_{2i-1} z'_{2i-1}\right]^{-1} \left[\sum_{i=1}^{t-1} \alpha_i z_{2i-1} z'_{1i}\right].$$

When $\alpha_i = 1$ for all i, this is just least squares. When α_i is increasing in i, this is a version of weighted least squares with a "forgetting factor" α_i that weights more recent observations more heavily.[6] See Ljung and Soderstrom [18] for recursive formulations of a variety of least squares algorithms, and for interpretations of them in terms of stochastic approximation and stochastic Gauss–Newton algorithms.

When D_1 is a proper subset of $\mathbb{R}^{n_1 \times (n_2)^3}$, the algorithm (4) deviates from weighted least squares because it invokes a "projection facility" (4b) that prevents the estimator from ever leaving a set determined by D_1. The projection facility has the effect of converting the algorithm for β_t into a version of least squares which ignores those observations, sequentially selected, which threaten to drive (β_t, R_t) outside of D_1. This projection facility can be interpreted as forcing the agents to ignore those observations that do not agree with their priors. Similar formulations have been used in the learning literature (Jordan [15], Anderson and Taylor [1], Woodford [29], Frydman [12]), in the adaptive estimation literature (Ljung and Soderstrom [18]), and in the econometrics literature (Hansen [14]). For many applications, if a projection facility is used, the hypotheses of Ljung's theorems are easy to verify. Below, we shall see that the structure of some economic problems implies a natural choice of D_1. In particular, in the investment under uncertainty example (example e, Section 4), the model is well defined only when β_t is restricted to lie in a certain bounded region.[7]

One role of the projection facility (4b) can be to guarantee that the coefficients of β_t remain in a region for which the model makes economic sense

[6] By choosing the initial β_{t-1} and R_{t-1} matrixes appropriately, schemes that have a kind of Bayesian interpretation can be accommodated. Here the qualification "kind of" is intended to concede our recognition of the points made by Bray and Kreps [5] and Bray and Savin [6]. To give a Bayesian interpretation to (4a), for example, we must grant that agents' prior is repeatedly wrong. To have a Bayesian interpretation (4a) requires that agents erroneously believe that they live in an environment governed by a constant coefficient linear difference equation.

[7] To facilitate comparison with Ljung [19], we have formulated our system so that it closely resembles his algorithm. This is why we require the projection facility to operate on the R_t's. However, the main ideas of Ljung are more widely applicable. It is often possible to relax bits and pieces of his assumptions. Thus, for all of the examples reported in Section 4, R_t can be shown to be bounded independently of the convergence of β_t, so that it is not necessary to project R_t into a compact set. Corollary 2 makes this point explicit for a class of examples.

and the operators $T(\beta)$, $A(\beta)$, $B(\beta)$, and $V(\beta)$ are well defined. Another role can be to guarantee that β stays in a region for which (3) with $\beta_t = \beta\ \forall t$ generates a covariance stationary stochastic process z_t. In this regard, we will utilize a set $D_s \subset \mathbb{R}^{n_1 \times n_2}$, which we define as

$$D_s = \{\beta \in \mathbb{R}^{n_1 \times n_2} \mid \text{the operators } T(\beta), A(\beta), B(\beta), \text{and } V(\beta) \text{ in (3) are well defined, and the eigenvalues of } \left[\begin{smallmatrix} 0, & T(\beta) \\ & A(\beta) \end{smallmatrix}\right] \text{ are less than unity in absolute value}\}.$$

We have assumed that u_t is a white noise with a covariance matrix that is time invariant. It follows that the time invariant version of (3) that emerges when we set $\beta_t = \beta \in D_s$ generates a covariance stationary process z_t whose second moment matrix $Ez_t z_t'$ is well defined. Then the moment matrix of the subvector z_{2t} is well defined and is a function of β, which we emphasize in the notation

$$Ez_{2t}z_{2t}' = M_{z_2}(\beta).$$

The system whose limiting behavior we want to analyze is formed by Eqs. (3) and (4). We proceed by applying the method of Ljung [19] to this system. The method associates with the system (3)–(4) an ordinary differential equation that is simpler to analyze than (3) and (4), but that almost surely mimicks the behavior of (β_t, R_t) as $t \to \infty$. For (3), (4) the associated differential equation is

$$(d/dt)\begin{bmatrix} \beta' \\ R \end{bmatrix} = \begin{bmatrix} R^{-1}M_{z_2}(\beta)[T(\beta) - \beta]' \\ M_{z_2}(\beta) - R \end{bmatrix}. \tag{5}$$

The differential equation (5) is obtained mechanically from (3)–(4) by fixing $\beta_t = \beta \in D_s$, $(\beta, R) \in D_1$, and $R_t = R$, and performing the following calculations. First, deduce the stationary distribution of z_t from (3) with $\beta_t = \beta$ and calculate $Ez_{2t}z_{2t}' = M_{z_2}(\beta)$. Second, replace α_t by its limiting value of unity within the brackets on the right side of (4a). Then obtain the right side of the differential equation (5) by simply taking expectations of the terms appearing after (α_t/t) on the right side of (4a) at the fixed value of (β, R).

We employ seven assumptions about the system (3)–(4).

(A.1) The operator T has a unique fixed point $\beta_f = T(\beta_f)$ which satisfies $\beta_f \in D_s$.

(A.2) T is twice differentiable and A, B, and V have one derivative in D_s.

(A.3) The covariance matrix $M_{z_2}(\beta_f)$ is nonsingular.

(A.4) For all t, $\alpha_t > 0$; α_t is increasing in t; $\alpha_t \to 1$ as $t \to \infty$; and $\limsup_{t \to \infty} t\,|\alpha_t - \alpha_{t-1}| = K < \infty$.

(A.5) The vector u_t consists of m stationary random variables; u_t is serially independent. Further, $E\,|u_{it}|^p < \infty$ for all $p > 1$, all $i = 1, ..., m$.

(A.6) There exists a subset Ω_0 of the sample space with $P(\Omega_0) = 1$, two random variables $C_1(\omega)$ and $C_2(\omega)$, and a subsequence $\{t_k(\omega)\}$ for which

$$|z_{2t_k}(\omega)| < C_1(\omega) \qquad \text{and} \qquad |R_{t_k}(\omega)| < C_2(\omega)$$

for all $\omega \in \Omega_0$ and all $k = 1, 2, ...$.

(A.7) Either

(1) Let $D_1 = D_2 = \mathbb{R}^{n_1 \times (n_2)^3}$. Given the set Ω_0 and the subsequence $t_k(\omega)$ named in (A.6), there exists a compact set $D' \subset D_s \subset \mathbb{R}^{n_1 \times n_2}$ such that $\beta_{t_k}(\omega) \in D'$ for all k and all $\omega \in \Omega_0$. Also, for any initial condition $(\beta(0), R(0))$ with $\beta(0) \in D'$ and $|R(0)| < C_2(\omega)$, trajectories that solve (5) never leave a closed subset of D_s.

or

(2) Let D_2 be closed, D_1 be open and bounded, and assume that $\beta \in D_s$ for all $(\beta, R) \in D_1$. Also, assume that the trajectories of (5) with initial conditions $(\beta(0), R(0)) \in D_2$ never leave a closed subset of D_1.

Assumptions (A.1)–(A.5) are on a different footing from assumptions (A.6)–(A.7). Members of the former group are in the nature of regularity conditions which are both easy to check and usually readily satisfied for the kinds of applications we have in mind. However, in some applications, assumptions (A.6) and (A.7.1) are more difficult to verify while (A.7.2) can be regarded as less natural to impose. We now briefly discuss each of these two groups of assumptions

Assumption (A.1) is made solely to simplify the presentation. It is evident from Ljung [19] and from the proofs below how our results would extend to a model with multiple stationary rational expectations equilibria. Assumption (A.2) could be relaxed but would have to be replaced by alternative conditions that are more cumbersome. Our applications all satisfy (A.2). Assumption (A.3) rules out the possibility that in the rational expectations equilibrium there are linear dependencies among elements of z_{2t}. Equilibria in which such dependencies are present can always be modeled, through appropriate redefinitions of state variables, by a system satisfying (A.3). Assumption (A.4) is stronger than we need; (A.4) permits agents to assign more weight to recent observations by setting $\alpha_t > \alpha_{t-1}$ for all t.

Assumption (A.5) could be weakened to permit the u_t's to have nonzero means and certain types of nonstationarity of the u's.

We now turn briefly to the restrictions (A.6) and (A.7). Assumption (A.6) is automatically satisfied if z_{2t} contains only exogenous, ergodic variables. Otherwise, (A.6) can be difficult to verify. Our treatment of example (e) below illustrates how (A.6) can be verified in one case in which z_2 contains some endogenous variables. Assumption (A.7.1) can be very difficult to verify, though at least in the case in which z_{2t} contains only exogenous variables, there appears hope that generally applicable methods can be devised (Kushner and Clark [16] describe some possible leads). Evidently, (A.7.2) is straightforward to verify. The main thing that has to be checked to verify (A.7.2) is that at or near the boundary δD_1 of D_1, the differential equation (5) has trajectories pointing toward the interior of D_1.

We now state the central proposition of this paper.

PROPOSITION 1. *Let* (β_t, R_t) *be given by the learning scheme* (4). *Assume* (A.1) *to* (A.6). *Let* D_A *be the domain of attraction of the unique equilibrium of the differential equation* (5) (*namely* (β_f, R_f)). *If*

(A.7.1) *is satisfied and* $D' \subset D_A$, *or*

(A.7.2) *is satisfied and* $D_1 \subset D_A$,

then $\beta_t \to \beta_f$ *almost surely as* $t \to \infty$.

Proof. We will first check the set of conditions B on page 554 of Ljung [19].

Assumptions B.1 and B.2 in Ljung are implied by our (A.5). B.3, B.4, and B.5 in Ljung are implied by the smoothness assumptions on T, A, B, V in our (A.2).

For B.6 in Ljung, we see that the following limits exist and are given by

$$\lim_{t\to\infty} E(R^{-1}[z_{2t-1}z'_{2t-1}[T(\beta)-\beta]' + z_{2t-1}u'_t V(\beta)']) = R^{-1}M_{z_2}(\beta)[T(\beta)-\beta]$$

$$\lim_{t\to\infty} E(z_{2t-1}z'_{2t-1} - (1/\alpha_t)R) = M_{z_2}(\beta) - R,$$

where β and R are fixed and z_{2t} is evaluated at solutions of the difference equation (3) with $\beta_t = \beta$.

Assumption B.7 is implied by (A.5) and (A.2), where the Lipschitz constants $\mathscr{K}_1$ and $\mathscr{K}_2$ in Ljung are, respectively, the norms of the first and second derivatives of

$$\begin{bmatrix} R^{-1}z_{2t}[z'_{2t}(T(\beta)-\beta)' + u_{t+1}V(\beta)] \\ z_{2t}z'_{2t} - R \end{bmatrix}$$

with respect to β, R, and z_{2t}.

Finally, assumptions B.8 to B.11 are implied by our (A.4).

If (A.7.1) is satisfied, then since $z_{1t} = T(\beta_t) z_{2t-1} + V(\beta_t) u_t$, it follows from (A.6) that there exists a subsequence of $\{t_k\}$ such that $|z_{1t_k}|$, $|z_{2t_k}|$, $|\beta_{t_k}|$, and $|R_{t_k}|$ are bounded along this subsequence; therefore (20) in Ljung [19] is satisfied, and we can apply his Theorem 1. In view of (A.1), it is clear that the differential equation (5) has only one equilibrium given by $(\beta_f, M_{z_2}(\beta_f))$, so that $P((\beta_t, R_t) \to (\beta_f, R_f)) = 1$, by Ljung's Theorem 1, which implies $P(\beta_t \to \beta_f) = 1$.

If (A.7.2) is satisfied, then all the assumptions of Theorem 4 in Ljung [19] are satisfied, and we arrive at the same conclusion. ∎

In applying Proposition 1, it is important to keep in mind the interaction between, on the one hand, the set D_A (the domain of attraction of the fixed point of the ordinary differential equation (5)) and, on the other hand, the set D' (when assumption (A.7.1) is invoked) or D_1 (when (A.7.2) is invoked). The convergence asserted in Proposition 1 hinges critically on two features of these sets. First, D' or D_2 must be contained in D_A under (A.7.1) or (A.7.2), respectively. Second, when (A.7.2) is invoked, the trajectories of the ordinary differential equation (5) that originate in D_2 must never leave D_1. This second condition can be checked by verifying that at the boundary of D_1, the trajectories of (5) point toward the interior of D_1, as illustrated in Fig. 1 for the case $n_1 = n_2 = 1$.

Proposition 1 does not cover the case depicted in Fig. 2, in which at some points on the boundary of D_1 trajectories of (5) point away from the interior of D_1, since this is ruled out by (A.7.2). In this case, even though

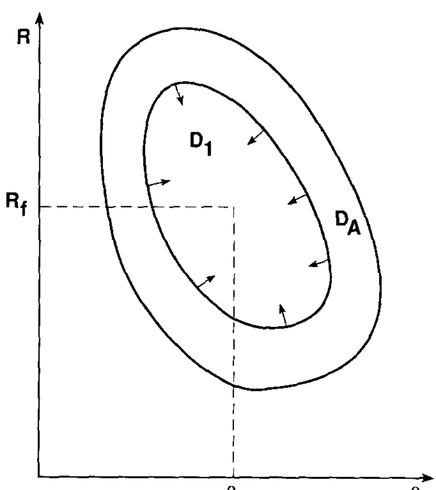

FIG. 1. Phase diagram of (5) for case in which on δD_1 all trajectories point toward interior of D_1.

$D_1 \subset D_A$, the possibility emerges that the algorithm (4) has a cluster point at the boundary δD_1. Heuristically, in mimicking the behavior of the differential equation (5), the algorithm "tries" to follow trajectories that leave D_1 once it is in the shaded area of Fig. 2, but is not allowed to do so by virtue of the projection facility (4b). This outcome is formalized in the following corollary to Proposition 1.

COROLLARY 1. *Assume that* (A.1)–(A.6) *are satisfied, that* $(\beta, R) \in D_1$ *implies* $\beta \in D_s$, *and that* D_1 *is open and bounded. Assume that* $D_1 \subset D_A$. *Then*

$$P(\beta_t \to \beta_f) + P(\beta_{t_k} \to (D_1 - D_2) \quad \textit{for a subsequence } \{t_k(\omega)\}) = 1.$$

In Corollary 1, "$\beta_{t_k} \to (D_1 - D_2)$" is the event that β_{t_k} is eventually close to some point in $D_1 - D_2$, not necessarily the same point for different t_k's. The subsequence t_k is permitted to depend on ω.

A version of Corollary 1 is stated by Ljung and Soderstrom [18, p. 163] which in turn is a corollary to Theorem 1 of Ljung [19].

In applying Proposition 1 for a given choice of D_1, an important step is verifying that at the boundary of D_1, the differential equation (5) has trajectories pointing into the interior of D_1. In general, this can be a demanding task that often requires resorting to numerical methods. There is an important class of cases, however, for which it is possible to verify analytically the required behavior of the trajectories of (5) at the boundary of D_1. These cases, in which z_2 is required to be exogenous in the sense that A

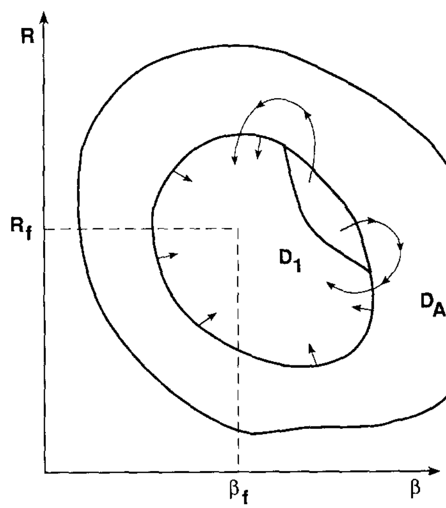

FIG. 2. Phase diagram of (5) for the case in which on δD_1 some trajectories point away from the interior of D_1 even though $D_1 \subset D_A$.

and B in (2) are independent of β, include the first four examples of Section 4. By means of describing this verification technique, we shall state another corollary to Proposition 1. The proof of this corollary will involve checking that the trajectories of (5) point toward the interior of an appropriate bounded set when the other conditions are satisfied; it will also involve some modifications in the proof of Theorem 4 of Ljung (strictly speaking Theorem 4 of Ljung does not apply because D_1 below is not bounded.)

Define H: $\mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{(n_1 \times n_2)^2}$ by letting $H(\beta)$ satisfy

$$\text{col}[T(\beta) - T(\beta_f)] = H(\beta)\,\text{col}[\beta - \beta_f],$$

where $H(\beta)$ is as given by the mean value theorem. In the particular case in which T is linear, $H(\cdot)$ is a constant function. Throughout the following corollary, $|\cdot|$ will denote the Euclidean norm.

COROLLARY 2. *Consider the algorithm* (3), (4a), (4b). *Choose any two numbers,* $0 < K' < K < \infty$. *Assume that*

(i) (A.1) *to* (A.5) *are satisfied.*

(ii) z_2 *is exogenous, so that* $E(z_2 z_2') = M_{z_2}(\beta) \equiv M$.

(iii) *The differential equation* $\dot{\beta} = T(\beta) - \beta$ *is globally stable in* $\mathbb{R}^{n_1 \times n_2}$.

(iv) $\exists \bar{\varepsilon} > 0$ *such that for all* $0 < \varepsilon < \bar{\varepsilon}$ *and all* β: $|\beta - \beta_f| = K$, *each eigenvalue of the matrix* $[I(1-\varepsilon) + \varepsilon H(\beta)]' \cdot [I(1-\varepsilon) + \varepsilon H(\beta)]$ *is less than* α^2, $\alpha < 1$, *in modulus.*

Then, taking

$$D_1 = \{(\beta, R) \in \mathbb{R}^{n_1 \times (n_2)^3} : |\beta - \beta_f| < K\}$$
$$D_2 = \{(\beta, R) \in \mathbb{R}^{n_1 \times (n_2)^3} : |\beta - \beta_f| \leqslant K'\}$$

and letting 4b *take the alternate form*

$$\left.\begin{array}{l} \beta_t = \text{some value in } \{\beta : |\beta - \beta_f| \leqslant K'\} \\ R_t = \bar{R}_t \end{array}\right\} \quad \text{if } (\bar{\beta}_t, \bar{R}_t) \notin D_1,$$

it follows that

$$\beta_t \to \beta_f \qquad a.s.$$

Proof. Take $\varepsilon^* > 0$ such that if $|R - M| \leqslant \varepsilon^*$ then $|R^{-1}M - I| < (1-\alpha)K/T_K$, where α is as in condition (iv) of this corollary, and

$$T_K = \max_{\{\beta : |\beta - \beta_f| \leqslant K\}} |T(\beta)| + |\beta|.$$

Define the sets

$$D_1^* = D_1 \cap \{(\beta, R): |R - M| < \varepsilon^*\}$$

$$D_2^* = D_2 \cap \left\{(\beta, R): |R - M| \leqslant \frac{\varepsilon^*}{2}\right\}.$$

These sets will play the role of D_1, D_2 in Theorem 4 of Ljung.

Next we show that at the boundary of D_1^* trajectories of (5) point toward the interior of the set D_1^*. Note that $\delta D_1^* = \{(\beta, R) \in \mathbb{R}^{n_1 \times (n_2)^3}: |\beta - \beta_f| = K$ or $|R - M| = \varepsilon^*\}$. Note that if $(\beta, R) \in \delta D_1^*$ then $|\beta - \beta_f| \leqslant K$ and $|R - M| \leqslant \varepsilon^*$. We need to show that given $(\bar{\beta}, \bar{R}) \in \delta D_1^*$, $\exists \bar{\bar{\varepsilon}} > 0$ such that for all $0 < \varepsilon < \bar{\bar{\varepsilon}}$,

$$\begin{bmatrix} \bar{\beta}' \\ \bar{R} \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{R}^{-1} M[T(\bar{\beta}) - \bar{\beta}] \\ M - \bar{R} \end{bmatrix} \in D_1^*.$$

This is equivalent to showing

$$|\bar{\beta} + \varepsilon \bar{R}^{-1} M[T(\bar{\beta}) - \bar{\beta}] - \beta_f| < K \qquad \text{and} \tag{$*$}$$

$$|\bar{R} + \varepsilon(M - \bar{R}) - M| < \varepsilon^*. \tag{$**$}$$

Condition ($**$) is trivial: so long as $\bar{\bar{\varepsilon}} < 1$, $|\bar{R} + \varepsilon(M - \bar{R}) - M| = (1 - \varepsilon)\,|\bar{R} - M| \leqslant (1 - \varepsilon)\varepsilon^* < \varepsilon^*$.

Now, for ($*$) take $\bar{\bar{\varepsilon}} = \min\{\bar{\varepsilon}, \frac{1}{2}\}$, where $\bar{\varepsilon}$ is as in condition (iv) of this corollary. If $0 < \varepsilon < \bar{\bar{\varepsilon}}$

$$\begin{aligned} \bar{\beta} + \varepsilon[T(\bar{\beta}) - \bar{\beta}] - \beta_f &= \bar{\beta} + \varepsilon[T(\bar{\beta}) - \bar{\beta}] - \beta_f - \varepsilon[T(\beta_f) - \beta_f] \\ &= (\bar{\beta} - \beta_f)(1 - \varepsilon) + \varepsilon[T(\bar{\beta}) - T(\beta_f)]; \end{aligned}$$

it follows that

$$\begin{aligned} \operatorname{col}[\bar{\beta} + \varepsilon[T(\bar{\beta}) - \bar{\beta}] - \beta_f] &= (1 - \varepsilon) \operatorname{col}(\bar{\beta} - \beta_f) + \varepsilon H(\bar{\beta}) \operatorname{col}(\bar{\beta} - \beta_f) \\ &= [I(1 - \varepsilon) + H(\bar{\beta})] \operatorname{col}(\bar{\beta} - \beta_f). \end{aligned}$$

Now we can use the fact that for any matrix A such that $A'A$ has all eigenvalues less than α^2 in absolute value, and any vector x, $|Ax| < \alpha\,|x|$ (see Noble and Daniel [23], Theorem 5.7). Therefore we conclude

$$|\bar{\beta} + \varepsilon[T(\bar{\beta}) - \bar{\beta}] - \beta_f| < \alpha\,|\bar{\beta} - \beta_f| = \alpha K, \tag{$***$}$$

where we have used condition (iv) in this corollary and $\bar{\bar{\varepsilon}} \leqslant \bar{\varepsilon}$.

Finally,

$$\begin{aligned}&|\bar{\beta}+\varepsilon R^{-1}M[T(\beta)-\beta]-\beta_f|\\&\quad=|\beta+\varepsilon[T(\beta)-\beta]+\varepsilon(R^{-1}M-1)[T(\beta)-\beta]-\beta_f|\\&\quad\leqslant \alpha K+\varepsilon\,|T(\bar{\beta})-\bar{\beta}|\cdot|R^{-1}M-I|\\&\quad\leqslant \alpha+\frac{1}{2}\,T_K\,\frac{(1-\alpha)K}{T_K}=\frac{1+\alpha}{2}\,K<K,\end{aligned}$$

where we have used the triangle inequality, $\bar{\bar{\varepsilon}}\leqslant\frac{1}{2}$, and $|R^{-1}M-I|<(1-\alpha)K/T_K$. Hence, $(*)$ is satisfied.

Now since the second part of (5) simply says that $\dot{R}=M-R$, letting $(\beta(t), R(t))$ be the trajectories of (5), we have that $R(t)\to M$ as $t\to\infty$. This plus condition (iii) of this corollary imply that $\beta(t)\to\beta_f$, and (5) is globally stable. Therefore, there exists a Lyapunov function $V\colon \mathbb{R}^{n_1\times(n_2)^3}\to[0,\infty)$ for (5) with the usual properties. Since D_1^* is bounded and we have shown that trajectories that start at δD_1^* point inside D_1^*, we can choose $V(\cdot)$ to satisfy

$$\begin{aligned}V(\beta, R)&\geqslant c_1 &&\text{for all } (\beta, R)\notin D_1^*\\V(\beta, R)&\leqslant c_2<c_1 &&\text{for all } (\beta, R)\in D_2.\end{aligned}$$

Finally, if (β_t, R_t) "attempted" to go out of D_1 (and, therefore, out of $D_1^*\subset D_1$) infinitely often (i.o.), then $(\beta_t, R_t)\in D_2^*$ i.o. This is because when $(\bar{\beta}_t, \bar{R}_t)\notin D_1$, $(\beta_t-\beta_f)\leqslant K'$ by the projection and, since $R_t\to M$, $|R_t-M|<\varepsilon$ for all t large enough. But if this were the case, $V(\beta_t, R_t)$ would go from a lower value c_2 to a higher value c_1 i.o., which by Step 3, Theorem 1 of Ljung [19] can only happen with zero probability.

Therefore, the projection is invoked finitely many times (with probability one); after the last time the projection is invoked, the algorithm behaves as if assumption (A.7.1) were satisfied, with $D'=D_1^*$, and Proposition 1 applies. ∎

It can be shown that a sufficient condition for (iv) in Corollary 2 is that $H(\beta)$ be symmetric and that all of its eigenvalues be less than zero for all β such that $|\beta-\beta_f|\leqslant K$. Also, note how the set D_1 restricts β_t to belong to a ball around β_f, and how R_t is never affected by the projection; D_1 and D_2 could be chosen arbitrarily large, and D_2 arbitrarily close to D_1. Sets with different shapes would work in many examples.

At this point, it is useful to recall the following results from the theory of ordinary differential equations. First, we represent Eq. (5) in terms of a state vector rather than the state matrix $[{}^{\beta}_{R}]$ that appears in (5). By stacking columns of matrices on each side of (5) we obtain

$$\frac{d}{dt}\begin{bmatrix}\text{col}(\beta)\\ \text{col}(R)\end{bmatrix} = g(\beta, R),$$

where col(β) is an $(n_1 \cdot n_2)$ vector obtained by stacking columns of β on top of each other and col(R) is an $(n_2^{(2)})$ vector obtained by stacking columns of R on top of one another. Now define

$$h(\beta, R) = \frac{d}{d(\text{col}\,\beta, \text{col}\,R)'}\, g(\beta, R).$$

If $h(\beta_f, R_f)$ has an eigenvalue with positive real part, then (5) is unstable. If all eigenvalues of $h(\beta_f, R_f)$ are negative in real part, then (β_f, R_f) is a locally stable rest point of (5).

These facts about ordinary differential equations are useful in interpreting the next two propositions. The first part of the proposition reaffirms a point made by Bray and Savin [6] that only a rational expectations equilibrium is a possible limit point of a least squares learning algorithm. The second part states a sense in which the stability of (5) is not only a sufficient but also a necessary condition for convergence.

PROPOSITION 2. *Assume that* (A.1) *to* (A.5) *are satisfied.*

(i) *Let* $\hat{\beta} \neq \beta_f$; *assume that* $M_{z_2}(\hat{\beta})$ *is positive definite and that* $\hat{\beta} \in \text{int}(D_2)$. *Then*

$$P(\beta_t \to \hat{\beta}) = 0.$$

(ii) *If* $h(\beta_f, R_f)$ *has one or more eigenvalues with strictly positive real part, then*

$$P(\beta_t \to \beta_f) = 0.$$

Proof. For part (i), we have

$$P(\beta_t \to \hat{\beta}) = P(\beta_t \to \hat{\beta} \text{ and } R_t \to M_{z_2}(\hat{\beta}))$$
$$+ P(\beta_t \to \hat{\beta} \text{ and } R_t \not\to M_{z_2}(\hat{\beta})).$$

The first probability on the right side is zero by the converse of Theorem 2 in Ljung [19]. That the second probability is also zero is shown in Lemma 1, Appendix II of this paper. Therefore $P(\beta_t \to \hat{\beta}) = 0$.

Part (ii) is shown by appealing again to Theorem 2 in Ljung and our Lemma 1, which is proved in Appendix II. ∎

Note that this proposition is obtained without invoking either (A.6) or (A.7). This means that if the sets D_1 and D_2 in (4a)–(4b) are set equal to $\mathbb{R}^{n_1 \times (n_2)^3}$, Proposition 2 remains true. We have the condition that $\hat{\beta}$ belongs to the interior because with D_1 unrestricted, the case depicted in Fig. 2 could arise.

We now describe a sense in which the limiting behavior of the differential equation (5) is determined by the simpler differential equation

$$\frac{d}{dt}\beta = T(\beta) - \beta. \tag{6}$$

We first note that if R_f is nonsingular, as assumed in (A.3), then (β_f, R_f) is a stationary point of (5) if and only if β_f is a stationary point of (6). This assertion follows directly from the definition of a stationary point.

The local stability of (6) is governed by the matrix

$$\mathcal{M} = \left.\frac{d(\operatorname{col}(T(\beta) - \beta))}{d \operatorname{col} \beta'}\right|_{\beta = \beta_f}.$$

The next proposition asserts that (5) and (6) share local stability properties.

PROPOSITION 3. (i) *$h(\beta_f, R_f)$ has $(n_2)^2$ repeated eigenvalues of -1.*

(ii) *Let A be the set of the (possibly repeated) remaining $(n_1 \times n_2)$ eigenvalues of $h(\beta_f, R_f)$. Let B be the set of all eigenvalues of $\mathcal{M}$. Then $A = B$.*

Proof. It is easy to check that

$$h(\beta_f, R_f) = \begin{bmatrix} (d/d \operatorname{col} \beta') \operatorname{col}[T(\beta_f) - \beta_f] & 0 \\ (d/d \operatorname{col} \beta') \operatorname{col} M_{z_2}(\beta_f) & -I \end{bmatrix} = \begin{bmatrix} \mathcal{M} & 0 \\ A_{21} & -I \end{bmatrix}$$

where we have used that $T(\beta_f) = \beta_f$ and $M_{z_2}(\beta_f) = R_f$ is invertible.

Clearly, any vector of the form $[{}^{0}_{m^*}]$, where 0 is an $n_1 \times n_2$ zero-vector and m^* has all elements except one equal to zero, forms an eigenvector with eigenvalue -1. Since there are $n_2 \times n_2$ such eigenvectors, we have part (i) of the proposition.

Now, let $\{\lambda, [{}^{m}_{m^*}]\}$ be one of the remaining pairs of eigenvalues and eigenvectors, where m has $n_2 \times n_1$ elements and m^* has $(n_2)^2$. If $m = 0$, then $[{}^{m}_{m^*}]$ would be a linear combination of the $n_2 \times n_2$ eigenvectors considered in the first part, which is not possible. Hence, the above formula for $h(\beta_f, R_f)$ gives $\mathcal{M}m = \lambda m$ for $m \neq 0$, and λ is an eigenvalue of $\mathcal{M}$. In this way we can find *all* $n_1 \times n_2$ eigenvalues of $\mathcal{M}$, so we have proved (ii). ∎

Proposition 3 implies that if all eigenvalues of $\mathscr{M}$ have real part strictly less than zero, then both (5) and (6) are locally stable. Under this circumstance, one can find sets D_1 and D_2 that satisfy (A.7.2), and for which there occurs the convergence asserted in Proposition 1.[8]

Proposition 3 also implies that if one eigenvalue of $\mathscr{M}$ has positive real part, then (5) and (6) are both unstable. Under this circumstance, there exists no choice of sets D_1 and D_2 for which there occurs convergence of β_t to β_f, except for the trivial choice $D_1 = \{\beta_f, R_f\}$.

3. Learning from Contemporary Data

The model (3), (4) shares with many examples in the literature the feature that the estimate β_t at t is a function of z_{1s}, z_{2s-1} for $s \leqslant t-1$. However, in many other applications it is desirable to alter a model to permit β_t at t to depend on data (z_{1t}, z_{2t-1}). This amendment makes β_t and z_t simultaneously determined, and complicates convergence proofs based on previous methods (e.g., see Bray and Savin [6] and Fourgeaud, Gourieroux, and Pradel [11]). In this section, we briefly describe how the apparatus of this paper would apply in such setups.

To accommodate contemporary information, we replace (4a) by

$$\begin{aligned} \bar{\beta}'_t &= \beta'_{t-1} + (\alpha_t/t)\, R_{t-1}^{-1} z_{2t-1}[z'_{1t} - z'_{2t-1}\beta'_{t-1}] \\ \bar{R}_t &= R'_{t-1} + (\alpha_t/t)[z_{2t}z'_{2t} - R_{t-1}/\alpha_t]. \end{aligned} \tag{4c}$$

Our system is now formed by (4c), (4b), and (3).

To ensure that the system (3), (4c), (4b) is well defined, we state another assumption:

(A.8) Let Ω_0 be the subset of the sample space defined in (A.6). For $\omega \in \Omega_0$, (3) and (4c), (4b) have a unique solution (β_t, R_t, z_t).

We now state the following proposition. As in the previous section, let D_A be the domain of attraction of the differential equation (5).

Proposition 4. *Assume* (A.1)–(A.6), (A.8). *If*

(A.7.1) *is satisfied and* $D' \subset D_A$, *or*

(A.7.2) *is satisfied and* $D_1 \subset D_A$,

then $\beta_t \to \beta_f$ *almost surely as* $t \to \infty$.

[8] Note that the differential equation (6) forces β at every point in time to be adjusted toward $T(\beta)$ by a small amount. By comparison, iterations on T of the kind studied by DeCanio [8] and Evans [9, 10] set $\beta_t = T(\beta_{t-1})$. Note that, in general, when iterations of this type converge, (6) is also stable. However, (6) can be stable when $\beta_t = T(\beta_{t-1})$ fails to converge, for example when some eigenvalues of $dT(\beta)/d\beta$ are less than -1 for all β.

This proposition can be proved by straightforward modification of the steps used by Ljung [19] to prove his Theorems 1 and 4. Versions of Corollaries 1 and 2 and Propositions 2 and 3 also hold. The upshot is that, provided (A.8) is satisfied, the same differential equations (5) and (6) govern the system (3), (4c), (4b) with contemporary learning, that govern the system (3), (4a), (4b) which imposes a one-period lag in agents' regressions.

4. Five Examples

The results of Sections 2 and 3 can be put to work by mapping a particular economic model into a system of the form (2) and (3), then verifying assumptions (A.1) to (A.7). To obtain a system of the form (2) and (3), one begins by writing down a set of equations that determine the equilibrium of an economic model. Often these will include linear "expectational difference equations", in which there appear some particular sets of agents' expectations about future variables.[9] For example, "Euler equations" are of this form. One completes the task by substituting elements of $\beta_t z_{2t-1}$ into these expectational difference equations, and solving for z_t in the form of (3).

This section describes five examples that illustrate how to apply our results. The examples are presented in order of increasing complexity in terms of the analysis needed to verify assumptions (A.6) and (A.7). The first four examples are important ones from the literature, while the fifth is one of our own making.

The first example involves learning only about variables exogenous to the agent, and is a system with no self-referential aspect. The second and third examples are ones in which agents influence the law of motion of the current value of an endogenous variable via their expectation about that current value, which is formed as a function of an exogenous variable. In the fourth example, agents influence the current value of an endogenous variable via their expectation about a *future* value of that variable, which is formed as a function of the current value of an exogenous variable.

The first four examples all share the property that z_{2t} includes only exogenous variables, so that $M_{z_2}(\beta)$ is independent of β, making Corollary 2 available. By contrast, the fifth example is one in which agents learn about an endogenous variable by regressing on lagged values of both exogenous and endogenous variables. This makes $M_{z_2}(\beta)$ a function of β and considerably complicates the analysis.

[9] See Charles Whiteman [28] for a discussion of linear "expectational difference equations" and how they arise in a variety of economic models.

a. *Estimation of Time Invariant Linear Stochastic Difference Equations*[10]

Consider the special case in which the law of motion is independent of the perceived law of motion. Let $T(\beta) = \gamma$, where γ is a square matrix of full rank whose eigenvalues are all strictly less than unity in modulus. Let $z_t = z_{1t} = z_{2t}$ and $V(\beta) = I$. Assume that u_t is an independently and identically distributed random vector whose covariance matrix has full rank and whose pth moments are finite. It follows that z_t is a covariance stationary process, and that $M_{z_2}(\beta) = Ez_t z_t'$ is independent of β. For this example, use Corollary 2 for arbitrary K, K', and note that $H(\bar{\beta}) = -I$ for all $\bar{\beta}$. Set $\bar{\varepsilon} = \frac{1}{2}$ in Corollary 2. Then it follows, for arbitrarily large values of the boundary parameter K used to define D_1, that Corollary 2 holds for this example. Therefore, $\beta_t \to \gamma$ almost surely as $t \to \infty$.

b. *A Model of Bray*

Margaret Bray [4] studied an example for which in our notation (3) takes the form

$$p_t = a + b\beta_t + \tilde{u}_t$$

and (4a) takes the form

$$\beta_t = \beta_{t-1} + \frac{1}{t} R_{t-1}^{-1} [p_{t-1} - \beta_t]$$

$$R_t = R_{t-1} + \frac{1}{t} [1 - R_{t-1}], \qquad R_0 = 1.$$

Our β_t is p_{t+1}^e in Bray's notation, where $\beta_t = p_{t+1}^e$ is the expected price, p_t is the price, and $\tilde{u}_t$ is an independently and identically distributed random process with zero mean and finite variance. To map this into our setup, we set $z_{1t} = p_t$, $z_{2t} = 1$, $u_t = \tilde{u}_t$,

$$z_t = \begin{bmatrix} p_t \\ 1 \end{bmatrix}, \qquad T(\beta) = a + b\beta,$$

$$V = 1, \qquad B(\beta) = 0, \qquad A(\beta) = [0, 1]'.$$

The ordinary differential equation (5) is

$$\frac{d}{dt} \beta = R^{-1} [a + (b-1)\beta]$$

$$\frac{d}{dt} R = 1 - R.$$

[10] This is a version of the problem studied by Mann and Wald [21].

Assumptions (A.1)–(A.6) are satisfied for this model. From the phase diagram of the differential equation (5) for Bray's model, shown in Fig. 3 for the case in which $b<1$, it follows that (A.7.2) is easy to satisfy. One can easily construct arbitrarily large sets D_1 containing β_f along whose boundary the trajectories of (5) point toward the interior of D_1. Figure 3 depicts the case for sets restricting β to satisfy $|\beta|<K$. Notice that Margaret Bray's algorithm in effect restricts R_t always to be unity.

Propositions 1–3 apply to this model, so that if $b<1$, then $\beta_t \to \beta_f$ almost surely. If $b>1$, then $P[\beta_t \to \beta_f]=0$.

c. *A Model of Bray and Savin*

A model of Bray and Savin [6] fits into our framework. In their notation, let the perceived law of motion for price in period t be $p_t = x_t' b_{t-1} + \omega_t$, while the actual price is determined by $p_t = x_t'(m + ab_{t-1}) + \tilde{u}_t$. Here $(x_t, \tilde{u}_t)$ is an independently and identically distributed vector of random variables with finite covariance matrix, and x_t is orthogonal to $\tilde{u}_s$ for all t and s. The vector x_t is an exogenous process of shifters to supply. The coefficient b_{t-1} is a least squares estimator of the perceived law of motion, based on data on p_s and x_s through $t-1$.

To map this model into our notation, let $z_t = (p_t, x_{t+1})'$, $z_{1t} = p_t$, $z_{2t} = x_{t+1}$, $\beta_t = b_{t-1}$, $T(\beta) = m + a\beta$, $A(\beta) = [0, 0]'$, $V(\beta) = B(\beta) = 1$, $u_t = [\tilde{u}_t, x_{t+1}]'$. The differential equation (5) for this model is

$$\frac{d}{dt}\begin{bmatrix}\beta\\ R\end{bmatrix} = \begin{bmatrix}R^{-1}m_x[(a-1)\beta + m]\\ m_x - R\end{bmatrix},$$

where $m_x = Ex_t x_t'$. In the case of a univariate x process, m_x is a scalar so that apart from the value for the constant and slope parameters, this differential equation system is identical with the one corresponding to (5) for

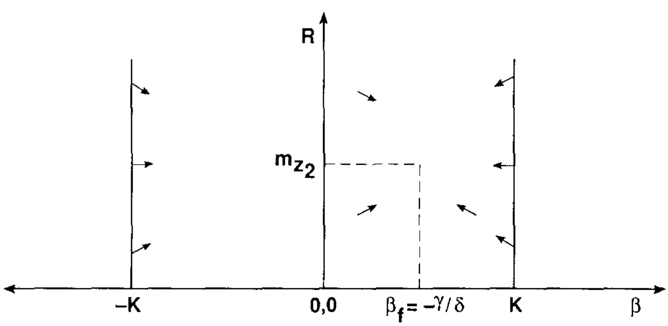

FIG. 3. The phase diagram of $(d/dt)\begin{bmatrix}\beta\\ R\end{bmatrix} = \begin{bmatrix}R^{-1}m_{z_2}(\gamma+\delta\beta)\\ m_{z_2}-\beta\end{bmatrix}$ when $\delta<0$. For Bray's model, $m_{z_2}=1$, $\gamma=a$, and $\delta=(b-1)$. For Bray and Savin's model with scalar x, $m_{z_2}=m_x$, $\gamma=m$, and $\delta=a-1$. For Fourgeaud, Gourieroux and Pradel's model, $m_{z_2}=m_x$, $\gamma=\rho$, and $\delta=(\lambda\rho-1)$.

the model of Bray analyzed in Section (4b). It follows when $a<1$ that qualitatively the phase diagram matches that of Fig. 3 (only now $\beta_f = m/(1-a)$ and $R_f = m_x$), and that our propositions apply to the Bray–Savin model in exactly the way they apply to the Bray model.

In the case of a multivariate x_t process, Corollary 2 applies to this model. Since $H(\beta) = I \cdot (a-1) < 0$, we can take $\bar{\varepsilon} = \frac{1}{2}$ and arbitrarily large $K' < K$. With the sets D_1, D_2 chosen as in Corollary 2, it follows that if $a<1$, then $\beta_t \to \beta_f = m/(1-a)$ almost surely as $t \to \infty$.

Note how the above argument could dispense with the assumption made by Bray and Savin that x_t is independently distributed through time. Also, by Proposition 3, if $a>1$, then $P(\beta_t \to \beta_f) = 0$.

d. *Hyperinflation or Stock Prices*

We consider a version of the model studied by Fourgeaud, Gourieroux, and Pradel [11]. The model is

$$\begin{aligned} y_t &= \lambda E_t y_{t+1} + x_t, && |\lambda| < 1,\ t \geqslant 1 \\ x_t &= \rho x_{t-1} + \varepsilon_t, && |\rho| < 1,\ t \geqslant 1,\ x_0 \text{ given,} \end{aligned} \tag{7}$$

where ε_t is a white noise with mean zero and $E\,\varepsilon_t x_{t-j}$ for $j>0$. At time t, the perceived law of motion for y_t is given by

$$y_t = \beta_t x_{t-1} + v_t,$$

where β_t is a least squares estimate based on data on y_s, x_{s-1} through $t-1$, and v_t is a least squares residual. Agents are assumed to form $E_t\, y_{t+1}$ in (7) according to $E_t\, y_{t+1} = \beta_t x_t$. We replace $E_t\, y_{t+1}$ by $\beta_t x_t$ in (7), and obtain

$$y_t = (\lambda\beta_t + 1) x_t = (\lambda\beta_t + 1)\,\rho x_{t-1} + (\lambda\beta_t + 1)\varepsilon_t.$$

With y_t interpreted as the price level and x_t as the money supply, the model is a version of Cagan's [7] model of hyperinflation studied by Sargent and Wallace [25]. With y_t interpreted as the price of a stock and x_t as its dividend, the model is a version of the "present value relation" (see Shiller [27] or LeRoy and Porter [17]).

This model fits into (3)–(4) with the following selections: $z_t = (y_t, x_t)'$, $z_{1t} = y_t$, $z_{2t} = x_t$, $T(\beta) = (\lambda\beta + 1)\rho$, $u_t = \varepsilon_t$, $V(\beta) = \lambda\beta + 1$, $B(\beta) = 1$, $A(\beta) = [0, \rho]$. Assumptions (A.1)–(A.5) all hold true. Here $\beta_f = \rho/(1-\lambda\rho)$, $R_f = m_x = Ex_t^2$. The ordinary differential equation (5) associated with (3) and (4) is

$$\frac{d}{dt}\begin{bmatrix} \beta \\ R \end{bmatrix} = \begin{bmatrix} R^{-1} m_x [(\lambda\rho - 1)\beta + \rho] \\ m_x - R \end{bmatrix}.$$

Apart from the interpretation of various slopes and constant terms, this equation is identical to the differential equation governing the models of Bray and of Bray and Savin (with univariate x_t). Once again, qualitatively the phase diagram is as in Fig. 3. Again, Propositions 1–3 apply. Using Corollary 2, since $|\lambda\rho| < 1$, we conclude $P(\beta_t \to \beta_f) = 1$ for all parameter values.

e. *Investment under Uncertainty with Learning*

We describe an example with a considerably more complex structure than any of the preceding four examples. The example is formed as an "adaptive" version of Sargent's [26, Chap. XIV] linear version of Lucas and Prescott's [20] model of investment under uncertainty. The model has agents learning about future values of an endogenous variable by regressing on lagged values of exogenous and endogenous variables. Associated with this model is a differential equation corresponding to (5) which is considerably more difficult to analyze than the ones associated with our previous examples.

A representative firm, of which there are N, faces the problem of maximizing

$$E \sum_{t=0}^{\infty} b^t \{p_t f k_t - w_t k_t - (d/2)(k_t - k_{t-1})^2\}$$

$$0 < b < 1, \qquad d > 0, \qquad f > 0 \tag{8}$$

subject to k_{-1} given, $K_{-1} = Nk_{-1}$ given, and the perceived laws of motion

$$\begin{aligned} w_t &= \rho w_{t-1} + u_t, \qquad |\rho| < 1 \\ p_t &= -A_1 f K_t, \qquad A_1 > 0 \\ K_t &= \beta_1 K_{t-1} + \beta_2 w_{t-1} + \omega_t, \end{aligned} \tag{9}$$

where u_t and ω_t are white noises that are orthogonal to w_{t-j}, K_{t-j} for $j > 0$. The model is formulated in deviations from means in order to dispense with a number of constant terms. Here p_t is the price of output, k_t is the capital stock of the representative firm, w_t is the rental rate of capital, $K_t = Nk_t$ is the aggregate capital stock in the industry, fk_t is output of the representative firm, and (fK_t) is output of the industry at t. The firm is supposed to act as if it knows the parameters of the laws of motion (9) with certainty. Under this circumstance, the firm's problem is a well posed optimal linear regulator problem.

The state for the firm at time t is (k_{t-1}, K_t, w_t), while the firm's control is $(k_t - k_{t-1})$ at t. The solution of the firm's problem is a time invariant decision rule that can be represented as

$$k_t = g_1 k_{-1} + g_2 w_t + g_3 K_t,$$

where g_1, g_2, g_3 are functions of the parameters A_1, f, d, b, ρ, β_1, and β_2. Multiplying by N, we have that the actual law of motion for K_t is

$$K_t = (1 - g_3 N)^{-1} [g_1 K_{t-1} + N g_2 \rho w_{t-1}] + N g_2 u_t, \qquad \text{or}$$

$$K_t = T_1(\beta_1, \beta_2) K_{t-1} + T_2(\beta_1, \beta_2) w_{t-1} + V(\beta_1, \beta_2) u_t. \tag{10}$$

Thus the firm's optimization problem and market clearing induce a mapping from a perceived law of motion for K_t given in (9) to the actual law of motion given in (10). The mapping $T(\beta)$ can be found by pursuing the sort of calculations described in Sargent [26, Chaps. IX and XIV]. The mapping is given by

$$T_1(\beta) = \frac{(1 - \beta_1 b)}{(1 - \beta_1 b + d^{-1} f^2 A_1 N)}$$

$$T_2(\beta) = \frac{(-d^{-1} N)}{(1 - \rho b)} \left\{ \frac{(1 - \beta_1 b + f^2 A_1 \beta_2 \rho b)}{(1 - \beta_1 b + d^{-1} f^2 A_1 N)} \right\} \rho. \tag{11}$$

To create a model of investment under uncertainty with learning, we assume, for simplicity, that agents know the true parameter ρ in the law of motion for w_t, but they learn about the law of motion for K_t. Then we can set $z_{1t} = K_t$, $z_{2t} = [K_t, w_t]$ in (2). Equation (3) takes the form

$$\begin{bmatrix} K_t \\ w_t \end{bmatrix} = \begin{bmatrix} T_1(\beta_t) & T_2(\beta_t) \\ 0 & \rho \end{bmatrix} \begin{bmatrix} K_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} V_1(\beta_t) \\ 1 \end{bmatrix} u_t. \tag{12}$$

The learning scheme is given by (4a)–(4b). We require that the sets D_1 and D_2 be such that if $(\beta, R) \in D_1$, then $|\beta_1| < b^{-.5}$ and $|\beta_2| < K$ for some arbitrary $K < +\infty$. The restriction that $|\beta_1| < b^{-.5}$ is required in order to assure that the firm's choice problem is well defined, an important requirement since this decision problem is used to define the T operator. For $|\beta_1| > b^{-.5}$, it occurs that the firm's objective function diverges to $-\infty$. We describe further possible restrictions on D_1 and D_2 below.

We proceed to apply our propositions to this model. Assumptions (A.1)–(A.5) are all satisfied. We shall discuss below how (A.6) and (A.7) can be satisfied by appropriate choice of D_1 and by restrictions on the u_t process. Corresponding to (5) we have the differential equation system

$$\frac{d}{dt}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = R^{-1}M_{z_2}(\beta)\begin{bmatrix}T_1(\beta)-\beta_1\\ T_2(\beta)-\beta_2\end{bmatrix}$$
$$\frac{d}{dt}R = M_{z_2}(\beta) - R. \tag{13a}$$

Corresponding to (6) we have the differential equation

$$\frac{d}{dt}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{bmatrix}T_1(\beta)-\beta_1\\ T_2(\beta)-\beta_2\end{bmatrix}. \tag{13b}$$

It can be verified that for all $|\beta_1| < b^{-.5}$, β_2 unrestricted, the differential equation (13b) has trajectories whose limit points are the unique rational expectations equilibrium. (In verifying this, it helps to start by noting that (13b) is recursive, since $T_1(\beta)$ depends only on β_1, and that for $|\beta_1| < b^{-.5}$, the ordinary differential equation $\dot{\beta}_1 = T_1(\beta_1) - \beta_1$ is stable because $dT_1(\beta_1)/d\beta_1 < 0$. Then it is also easy to see that for fixed β_1, $dT_2(\beta)/d\beta_2 < 0$, and β_2 also converges.)

From the stability of (13b) about the rational expectations equilibrium (β_f), it follows from Proposition 3 that there exist *some* nontrivial sets D_1 and D_2 for which assumption (A.7.2) can be verified. Of course, Proposition 3 provides no indication about how big these sets can be. To verify that a given choice of sets D_1 and D_2 will work, one has to verify the condition that at the boundary of D_1, all trajectories of the differential equation (5) point toward the interior of D_1. For a model as complicated as the present one, this condition seems very hard to verify analytically, because of the presence of the operator $M_{z_2}(\beta)$ on the right side of (5). However, for a given choice of D_1 and given parameter values, this condition could readily be verified by numerically solving for trajectories of the differential equation (5) at a grid of points on the boundary of D_1.

With a D_1 chosen to satisfy (A.7.2), in order that Proposition 1 apply, we have also to verify that (A.6) is satisfied. The fact that a zero appears in the (2.1) element of the matrix $\left[\begin{smallmatrix}T_1(\beta) & T_2(\beta)\\ 0 & \rho\end{smallmatrix}\right]$ appearing in Eq. (12) makes it possible to establish that

$$|R_t| \leqslant \frac{\sum_{i=1}^{t} |z_i z_i'|}{t} < C < \infty$$

almost surely for all t large enough, where C is a fixed constant (a full account of this is given in Appendix III). It follows that $|z_t| < C$ infinitely often with probability one, and (A.6) is satisfied.

We have thus described a list of conditions on D_1 which satisfy the hypotheses of Proposition 1.

5. Conclusions

We have described a general framework for analyzing the convergence of least squares learning in some self-referential linear models. Our framework provides an alternative, and in some respects simpler method of characterizing convergence for several models that have appeared in the literature. Our framework permits us to strengthen some results in the literature by dropping certain serial independence assumptions and by characterizing conditions under which convergence will not obtain. The framework also permits us to analyze models in which some of the regressors used by the agents are endogenous.

Our results are achieved by recognizing that the class of models that we study satisfies the regularity conditions used by Ljung [19], who showed conditions under which an ordinary differential equation governs the almost sure convergence of a system of stochastic difference equations. For our models, the ordinary differential equation involves the mapping $T(\beta)$ from perceived to actual laws of motion of the system. We also show that the convergence of least squares estimates to a rational expectations equilibrium bears an intimate relationship to the stability of the differential equation $(d/dt)\beta = T(\beta) - \beta$. This relationship provides a unifying and interpretable criterion for convergence that is applicable across a range of models.

A drawback of our results is that the stability of $(d/dt)\beta = T(\beta) - \beta$ assures only the local convergence of least squares, as indicated in our Proposition 3. The sufficient conditions for global convergence stated in Proposition 1 involve the behavior of a larger ordinary differential equation, which can be much more difficult to analyze in cases in which agents include endogenous variables among their regressors. One would like a stronger and more global implication about convergence to flow from stability of the "small" ordinary differential equation $d\beta/dt = T(\beta) - \beta$. A stronger result might be obtained in the future, perhaps by further restricting the mapping $T(\beta)$, and therefore also the mapping $M_{z_2}(\beta)$ which appears in the larger ordinary differential equation.

Another drawback of our approach, inherited from Ljung, is the need to verify certain boundedness assumptions ((A.6) and (A.7)), which can be troublesome in some contexts. Further research might be directed at providing more general and direct ways of verifying these conditions.[11]

The framework in this paper fails to apply to models having either hidden state variables or private information, some of which have been used in studies of convergence of least squares learning (Bray [3], Frydman [12]). In a sequel, we extend our framework to handle such cases.

APPENDIX I

Here we state the convergence theorems of Ljung [19]. This appendix is mostly a transcription of parts of that paper. We will use Ljung's equation numbers, preceded by the sign "I," to facilitate comparison.

Consider the algorithm

$$x_t = x_{t-1} + \gamma(t)\, Q(t; x_{t-1}, z_{t-1}), \tag{I-1}$$

where x_t is an n-dimensional vector; z_t is an m-dimensional state vector observed at time t; Q is a function mapping $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$; and $\gamma(t)$ is a sequence of positive (nonrandom) scalars. The process for the state vector z_t is given by

$$z_t = F(x_t) z_{t-1} + G(x_t) e_t, \tag{I-2}$$

where e_t is an r-dimensional vector of serially independent random variables, $F: \mathbb{R}^n \to \mathbb{R}^{m \times m}$, and $G: \mathbb{R}^n \to \mathbb{R}^{m \times r}$. Note that the process z_t is allowed to depend on x_t. To map this into our framework, x_t, $\gamma(t)$, z_t, and e_t in this appendix correspond to col(β_t), α_t/t, z_t, u_t of the paper, and Eqs. (I-1), (I-2) correspond to Eqs. (4) and (3). (In fact (4) corresponds to (I-1) only when $D_1 = D_2 = \mathbb{R}^{n_1 \times (n_2)^3}$). Let $\bar{z}_t(x)$ be the stochastic process given by

$$\bar{z}_t(x) = F(x)\, \bar{z}_{t-1}(x) + G(x) e_t \qquad \bar{z}_0(x) = 0,\ x \in R^n \tag{I-15}$$

[11] With the results currently available in the literature on adaptive control, it is not possible to derive the asymptotic distribution for our algorithm. Ljung and Soderstrom display the asymptotic distribution of algorithms that are contructed in order to find the zero of an appropriate loss function (see Chap. 4 of Ljung and Soderstrom [18]). In our case, the natural loss function to use would be the norm of the prediction error; but, since the mapping T enters the expression for z_t (the variable that is predicted), the term multiplying (α_{t-1}/t) in Eq. (4) is not the derivative of the loss function. That derivative would involve a term containing $\partial T/\partial \beta$. In the least squares case the algorithm is adjusted along the direction $z_{2t-2} z'_{2t-2}$, while for a model that is non-linear in β_t the best direction may be given by replacing $z_{2t-2} z'_{2t-2}$ with $H_{t-2} z_{2t-2}$, where H_t involves the derivative of T in the appropriate way. In this way, our algorithm is related to the pseudo-linear regression discussed by Ljung and Soderstrom.

and let $v(t, \lambda, c)$, where $0 < \lambda < 1$, and c is a positive constant, be given by

$$v(t, \lambda, c) = \lambda v(t-1, \lambda, c) + c\,|e(t)|, \quad v(0, \lambda, c) = 0. \tag{I-16}$$

Let $\mathscr{B}(\bar{x}, e) = \{x: |x - \bar{x}| < \rho\}$. Let D_R be an open, connected subset of D_s, D_s defined as in Section 2 of this paper. The following conditions are assumed to hold for all $x \in D_R$.

B.1. $e(\cdot)$ is a sequence of independent random variables (not necessarily stationary or with zero means).

B.2. $E\,|e(t)|^p$ exists and is bounded in t for each $p > 1$.

B.3. The function $Q(t, x, z)$ is Lipschitz continuous in x and z: $|Q(t, x_1, z_1) - Q(t, x_2, z_2)| < \mathscr{K}_1(x, z, \rho, v)\{|x_1 - x_2| + |z_1 - z_2|\}$ for $x_i \in \mathscr{B}(x, \rho)$ for some $\rho = \rho(x) > 0$ where $x \in D_R$; $z_i \in \mathscr{B}(z, v)$, $v \geqslant 0$.

B.4. $|\mathscr{K}_1(x, z_1, \rho, v_1) - \mathscr{K}_1(x, z_2, \rho, v_2)| \leqslant \mathscr{K}_2(x, z, \rho, v, w) \cdot \{|z_1 - z_2| + |v_1 - v_2|\}$ for $z_i \in \mathscr{B}(z, w)$ and $v_i \in \mathscr{B}(v, w)$.

B.5. $A(\cdot)$ and $B(\cdot)$ are Lipschitz continuous in D_R.

B.6. $\lim_{t \to \infty} EQ(t, \bar{x}, \bar{z}_t(\bar{x}))$ exists for $\bar{x} \in D_R$ and is denoted by $f(\bar{x})$. The expectation is over $e(\cdot)$.

B.7. For $x \in D_R$, the random variables $Q(t, x, \bar{z}_t(x))$, $\mathscr{K}_1(x, \bar{z}_t(x), \rho(x), v(t, \lambda, c))$, and $\mathscr{K}_2(x, \bar{z}_t(x), \rho(x), v(t, \lambda, c), v(t, \lambda, c))$ have bounded p-moments for all $p > 1$ and all $\lambda < 1$, $c < \infty$. Here $\bar{z}(x)$ and $v(\cdot, \lambda, c)$ are the random variables defined by (I-15) and (I-16).

B.8. $\sum_1^\infty \gamma(t) = \infty$.

B.9. $\sum_1^\infty \gamma(t)^p < \infty$ for some p.

B.10. $\gamma(\cdot)$ is a decreasing sequence.

B.11. $\lim_{t \to \infty} \sup[1/\gamma(t) - 1/\gamma(t-1)] < \infty$.

Consider the differential equation

$$\frac{d}{dt}\, x(t) = f(x(t)), \tag{I-19}$$

where f has been defined in B.6.

THEOREM 1 (Ljung [19]). *Consider the algorithm* (I-1), (I-2), *subject to assumptions* B. *Let* $\bar{D} \subset D_R$ *be compact, and assume that trajectories of* (I-19) *that start in* $\bar{D}$ *never leave a closed set* $\bar{D}_R \subset D_R$ *for* $t > 0$. *Assume that*

there is a random variable C *such that*

$$x_t \in \bar{D} \text{ and } |z_t| < C \text{ infinitely often with probability one.} \tag{I-20}$$

the differential equation (I-19) *has an invariant set*

$$D_c = \{x: f(x) = 0\} \text{ with domain of attraction } D_A \supset \bar{D}. \tag{I-21}$$

Then $x_t \to D_c$ *with probability one as* $t \to \infty$.

Ljung also provides an alternative algorithm, with a "projection facility." Let $D_2 \subset D_1 \subset D_s$,

$$\begin{aligned} \bar{x}_t &= x_{t-1} + \gamma(t)\, Q(t; x_{t-1}, z_{t-1}) \\ x_t &= \bar{x}_t \qquad \text{if } \bar{x}_t \in D_1 \\ x_t &= \text{some value in } D_2 \qquad \text{if } \bar{x}_t \notin D_1, \end{aligned} \tag{I-28}$$

and

$$\begin{aligned} z_t &= F(x_t)\, z_{t-1} + G(x_t) e_t \qquad \text{if } \bar{x}_t \in D_1 \\ z_t &= \text{a value in a given compact subset of } \mathbb{R}^m \qquad \text{if } \bar{x}_t \notin D_1. \end{aligned} \tag{I-29}$$

THEOREM 4 (Ljung [19]). *Consider the algorithm* (I-28), (I-29) *subject to assumptions* B. *Let* $D_1 \subset D_R$ *be an open bounded set containing the compact set* D_2. *Let* $\tilde{D} = D_1 \backslash D_2$ (D_1 *"minus"* D_2). *Assume that* $D_2 \subset D_A$, *with* D_A *defined as in Theorem* 1. *Suppose that there exists a twice differentiable function* $U(x) \geqslant 0$, *defined in a neighborhood of* $\tilde{D}$ *with properties*

$$\sup_{x \in \tilde{D}} U'(x) < 0, \tag{I-30}$$

$$\begin{aligned} U(x) &\geqslant C_1 \qquad \text{for } x \notin D_1 \\ U(x) &\leqslant C_2 < C_1 \qquad \text{for } x \in D_2. \end{aligned} \tag{I-31}$$

Then Theorem 1 *holds without assumption* (I-20).

Equations (I-30), (I-31) hold if the trajectories of (I-19) do not intersect the boundary of D_1 "outwards".

APPENDIX II

The following lemma is used in proving our Proposition 2.

LEMMA 1. *Assume that* z_t *is given by* (3) *and that* (A.2), (A.4), *and* (A.5) *are satisfied. Then*

$$P(\beta_t \to \hat{\beta} \text{ and } R_t \not\to M_{z_2}(\hat{\beta})) = 0$$

for any $\hat{\beta} \in D_s$.

Proof. We will first check that for any $\omega \in \Omega$ such that $\beta_t(\omega) \to \hat{\beta}$, and ω outside a set of probability zero, $\exists \{t_k\}$ such that $|z_{t_k}(\omega)| \leqslant C(\omega)$ for some random variable $C(\omega) < \infty$.

Note that we can choose $\bar{\varepsilon} > 0$ such that if $|\beta_k - \hat{\beta}| < \bar{\varepsilon}$ for all $k = n, ..., t$, then

$$\left| \prod_{k=n}^{t} T(\beta_f) \right| < C_1 \left(\frac{1+\lambda}{2} \right)^{t-n} \equiv C_1 (\bar{\lambda})^{t-n} \qquad \text{and}$$
$$|V(\beta_k)| \leqslant C_2,$$

where C_1, C_2 are fixed constants, $\lambda < 1$ is the maximum eigenvalue of $T(\hat{\beta})$ (in modulus). The proof of this uses continuity of $T(\cdot)$ (see I.11 in Ljung [19]).

Now, given $\omega \in \Omega$, if $\beta_t(\omega) \to \hat{\beta}$, $\exists n$ such that $\forall t > n$ the above inequalities are valid; also, since

$$z_t(\omega) = \prod_{i=n}^{t} T(\beta_i)\, z_n(\omega) + \sum_{i=n}^{t} \left[\prod_{\tau=i}^{t-1} T(\beta_\tau) \right] V(\beta_i)\, u_i(\omega),$$

we have

$$|z_t(\omega)| \leqslant C_1\, |z_n(\omega)| \cdot \bar{\lambda}^{t-n} + C_2 C_1 \left[\sum_{i=n}^{t} \bar{\lambda}^{t-i}\, |u_i(\omega)| \right].$$

We can write

$$\sum_{i=n}^{t} \bar{\lambda}\, |u_i| \equiv x_t = \bar{\lambda} x_{t-1} + |u_i| \qquad \forall t > n \qquad \text{with } x_n = |u_n|.$$

Since u has bounded pth moments, and $\bar{\lambda} < 1$, we have that x_n is ergodic, and we conclude that for some subsequence $\{t_k\}$ and some constant $C_3 < \infty$,

$$x_{t_k}(\omega) \leqslant C_3 \qquad k = 1, 2, ...,$$

with probability one. Hence

$$|z_{t_k}(\omega)| \leqslant |z_n(\omega)|\, C_1 + C_2 C_1 C_3, \qquad k = 1, 2, ...$$

This shows that z_t is bounded i.o.

Now, consider $0 < \rho < \bar{\rho}$, and n' such that if $k > n'$, $|\beta_k - \hat{\beta}| < \rho$. Having shown that z_t is bounded infinitely often, we can apply step 2 of Lemma 1 in Ljung [19, p. 565], and we have

$$|z_t - \bar{z}_t| < \bar{\lambda}^{t-n'} v(n', \lambda, c) + \rho v(t, \lambda, c) \equiv B_t,$$

where $\bar{z}_t$ is defined as $\bar{z}_t = T(\hat{\beta})\,\bar{z}_{t-1} + V(\hat{\beta})u_t$, $t>0$, $\bar{z}_0 = 0$ and where $v(s, \bar{\lambda}, c) = \bar{\lambda} v(s-1, \bar{\lambda}, c) + c\,|u_s|\; v(0, \bar{\lambda}, c) = 0$ for some constant c.

Since $\bar{z}$ and v are linear processes, we know that with probability one,

$$\frac{\sum^t |z_i|^2}{t} \to K_1 \quad \text{and} \quad \frac{\sum^t v(i, \bar{\lambda}, c)^2}{t}$$

$$\to K_2(c) \qquad \text{as } t \to \infty, \text{ for some } K_1,\ K_2(c) < \infty.$$

Letting $\bar{R}_t = (\sum_i^t z_{2i-1} z'_{2i-1})/t$, we have

$$|R_t - \bar{R}_t| \leqslant \frac{\sum_i^t |z_i z'_i - \bar{z}_i z'_i|}{t} \leqslant \frac{2\sum_i^t (|\bar{z}_i| + B_i)(B_i)}{t}$$

$$\leqslant 2\left(\frac{\sum^t (|\bar{z}_i| + B_i)^2}{t}\right)^{1/2} \left(\frac{\sum^t B_i^2}{t}\right)^{1/2}, \qquad \text{(II-1)}$$

where we have used the triangle inequality, the mean value theorem (which implies $|z_i z'_i - \bar{z}_i \bar{z}_i| \leqslant 2(|\bar{z}_i| + B_i) B_i$), and the Cauchy–Schwartz inequality.

Now,

$$\limsup_{t\to\infty} \frac{\sum_i^t (B_i)^2}{t} \leqslant \limsup_{t\to\infty} \left(2\left(\frac{\sum_i^t \lambda^{i-n}}{t}\, v(n', \bar{\lambda}, c)\right)^2 + 2\rho^2 \left(\frac{\sum^t v(i, \bar{\lambda}, c)}{t}\right)^2\right)$$

$$= 2 + 2\rho^2 (K_2)^2$$

independently of n' (which depends on ρ but not on t).

Using this, and taking the lim sup in both sides of Eqs. (II-1),

$$\limsup_{t\to\infty} |R_t - \bar{R}_t| \leqslant 2\,|2((K_1)^2 + (K_2)^2)^{1/2}\,(2(K_2)^2)^{1/2}$$

$$\leqslant 4((K_1)^2 + (K_2)^2)^{1/2}\, K_2 \rho.$$

Since this holds for any $\rho > 0$, we can let $\rho \downarrow 0$ and conclude that the above limsup is zero, so that $\lim_{t\to\infty} |R_t - \bar{R}_t| = 0$. Since, by ergodicity of $\bar{z}$, $\bar{R}_t \to M_{z_2}(\hat{\beta})$, we have $\lim_t \to \infty\ R_t = M_z(\hat{\beta})$.

This holds for any ω where β_t goes to $\hat{\beta}$ except for $\omega \in \Omega_0$, where Ω_0 is the union of several sets of probability zero for which convergence of averages may not hold. Hence $\{\omega \in \Omega : \beta_t(\omega) \to \hat{\beta}$ and $R_t \not\to M_z(\hat{\beta}\} \subset \Omega_0$, and we have our conclusion.

APPENDIX III

Here we show that for the example in section 4e, with probability one, there exists a t' such that if $t > t'$

$$\frac{1}{t} \sum_{i=0}^{t} |z_i z_i'| < C < \infty,$$

where C does not depend on the realization $\omega \in \Omega$.

By recursive substitution in the law of motion for z_t, we can write

$$z_t = A_{t,0} z_0 + \sum_{i=0}^{t} A_{t-1,i} V(\beta_i) u_i, \tag{III-1}$$

where

$$A_{t,i} = \prod_{j=i}^{t} \begin{bmatrix} T_1(\beta_j) & T_2(\beta_j) \\ 0 & \rho \end{bmatrix},$$

where z_t, T_1, T_2, V, u_t are as in section 4e.

The key to the proof will be that the norm of $A_{t,i}$ decays exponentially as t increases (i.e., is of exponential order less than one). The crucial property of the model that will allow us to show this is the zero that appears in the coefficients of z_{t-1} in the law of motion for z_t.

In section e, we have discussed how D_1 must be chosen so that $\forall(\beta, R) \in D_1$, $|\beta_1| < b^{-.5}$ and $|\beta_2| < K_1 < \infty$ for arbitrary K_1. If this is the case,

$$|T_1(\beta)| < \lambda < 1, \qquad |T_2(\beta)| < K_2 < \infty \qquad \forall(\beta, R) \in D_1. \tag{III-2}$$

After some algebra, it is possible to show

$$A_{t,\tau} = \begin{bmatrix} \prod_{i=\tau}^{t} T_1(\beta_i) & \sum_{i=\tau}^{t} T_2(\beta_i)\, \rho^{i-\tau} \prod_{j=i+1}^{t} T_1(\beta_j) \\ 0 & \rho^{t-\tau-1} \end{bmatrix}.$$

Let $\bar{\lambda} = \max\{\rho, \lambda\}$. By (III-2), the diagonal elements of $A_{t,\tau}$ are bounded in absolute value by $\bar{\lambda}^{t-\tau-1}$, and the (1, 2) element is bounded by $K_2(t-\tau)\,\bar{\lambda}^{t-\tau}$. Here $|T_2(\beta_i)| < K_2 < \infty$.

Now, choose $\bar{\bar{\lambda}} > \bar{\lambda}$, $\bar{\bar{\lambda}} < 1$. Since $n(\bar{\lambda}/\bar{\bar{\lambda}})^n \to 0$ as $n \to \infty$, there exists a positive, finite number c' such that $n\bar{\lambda}^n < c'\bar{\bar{\lambda}}^n$ for all n. Choosing $c' \geq K_2$, we have that all elements of $A_{t,\tau}$ are less than $c'\bar{\bar{\lambda}}^{t-\tau}$ in absolute value, and

$$|A_{t,\tau}| < c'\bar{\bar{\lambda}}^{t-\tau} \qquad \forall t, \tau. \tag{III-3}$$

Taking the norm of both sides of (III-1) and using (III-3),

$$|z_t| \leqslant |z_0|\, c' + K_3 c' \sum_{i=0}^{t} \bar{\lambda}^{t-i}\, |u_i| \qquad \text{for all } t,$$

where $K_3 < \infty$ is such that $|V(\beta)| < K_3$, $\forall(\beta, R) \in D_1$. Now, letting $x_t = \sum_{i=0}^{t} \bar{\lambda}^{t-i}\, |u_i|$, we see that

$$x_t = \bar{\lambda} x_{t-1} + |u_t| \qquad t > 0,$$

$$x_0 = 0.$$

Since $|u_t|$ has bounded pth moments, we have $(\sum^t x_i^2)/t \to K_4 < \infty$ as $t \to \infty$, with probability one,

$$\frac{\sum_{i=0}^{t} |z_t|^2}{t} \leqslant 2(|z_0|\, c')^2 + 2(K_3 c')^2 \cdot \frac{\sum^t x_i^2}{t}$$

and with probability one there is a t' such that if $t > t'$

$$|R_t| \leqslant 2\, |z_0|^2\, c'^2 + 2(K_3 c')^2\, (K_4 + 1) < \infty. \qquad \text{(III-4)}$$

(Note that t' depends on the realization $\omega \in \Omega$.)

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